Substitution

For indefinite integrals

$$\int f(g(x))g'(x)dx = \int f(u)du = F(u) + C = F(g(x)) + C$$

where $u = g(x)$, $du = g'(x)dx$, and $F(x)$ is an antiderivative of $f(x)$.

For definite integrals

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

where $u = g(x)$, $du = g'(x)dx$.

Integration By Parts (IBP)

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

Using the substitutions

$$\begin{array}{ll} u = f(x) & dv = g'(x)dx \\ du = f'(x)dx & v = g(x) \end{array}$$

we get the formula

$$\int udv = uv - \int vdu$$

IBP Tip: Typically, we want to choose u and dv so that $\int vdu$ is easier to integrate. When deciding how to choose u, consider the following priority list:

L-logarithms (e.g., $\ln(x)$)
I-inverse trig functions (e.g., $\arcsin(x)$)
A-algebraic functions (e.g., x^3)
T-trigonometric functions (e.g., $\sin(x)$)
E-exponentials (e.g., e^x)

Notice that the list of functions for u goes from functions that are difficult to integrate to functions that are easier to integrate since we need to know how to integrate our choice for dv.

Partial Fractions

Given the integral $\int \frac{P(x)}{Q(x)}dx$ where $P(x)$ and $Q(x)$ are polynomials such that the degree of $P(x)$ is smaller than the degree of $Q(x)$ $\left(\text{i.e., } \frac{P(x)}{Q(x)} \text{ is a proper rational function}\right)$, factor $Q(x)$ and find the partial fraction decomposition of $\frac{P(x)}{Q(x)}$ to get a sum of simpler integrals. There are 4 cases for the decomposition depending on how $Q(x)$ factors.

Case 1: If $Q(x)$ has a product of distinct linear factors

$$(a_1x + b_1)(a_2x + b_2)...(a_kx + b_k)$$

then these factors correspond to the following expression in the decomposition

$$\frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + ... + \frac{A_k}{a_kx + b_k}$$

where $A_1, ..., A_k$ are constants.

Case 2: If $Q(x)$ has a repeated linear factor, $(ax+b)^k$ where $k > 1$, then the corresponding expression in the decomposition is

$$\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + ... + \frac{A_k}{(ax + b)^k}$$

where $A_1, ..., A_k$ are constants.

Case 3: If $Q(x)$ has an irreducible quadratic factor, $ax^2 + bx + c$, that does not repeat, then the corresponding expression in the decomposition is

$$\frac{Ax + B}{ax^2 + bx + c}$$

where A and B are constants.

Case 4: If $Q(x)$ has a repeated irreducible quadratic factor, $(ax^2 + bx + c)^k$ where $k > 1$, then the corresponding term in the decomposition is

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + ... + \frac{A_kx + B_k}{(ax^2 + bx + c)^k}$$

where A_i and B_i are constants for $1 \leq i \leq k$.

If $\frac{P(x)}{Q(x)}$ is not a proper rational function (i.e., the degree of $P(x)$ is greater than or equal to the degree of $Q(x)$), then use polynomial long division on $\frac{P(x)}{Q(x)}$ to find

$$P(x) = Q(x)q(x) + r(x)$$

where $q(x)$ is the quotient and $r(x)$ is the remainder (and hence, the degree of $r(x)$ is less than the degree of $Q(x)$). We get then that

$$\frac{P(x)}{Q(x)} = q(x) + \frac{r(x)}{Q(x)}$$

and if necessary, since $\frac{r(x)}{Q(x)}$ is a proper rational function, we can find its partial fraction decomposition.

Trig Substitution

For integrals involving the term $\sqrt{a^2 - x^2}$, try the substitution $x = a\sin(\theta)$ and use the identity $1 - \sin^2(\theta) = \cos^2(\theta)$.

For integrals involving the term $\sqrt{a^2 + x^2}$, try the substitution $x = a\tan(\theta)$ and use the identity $1 + \tan^2(\theta) = \sec^2(\theta)$.

For integrals involving the term $\sqrt{x^2 - a^2}$, try the substitution $x = a\sec(\theta)$ and use the identity $\sec^2(\theta) - 1 = \tan^2(\theta)$.