Definite Integral

Let $f(x)$ be a continuous function over the interval $[a, b]$. Partition $[a, b]$ into $n$ subintervals of width $\Delta x = \frac{b-a}{n}$ and endpoints $x_0 = a, x_1, ..., x_n = b$. Let $x_i^* \in [x_{i-1}, x_i]$. Then

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^n f(x_i^*)\Delta x$$

Indefinite Integral

$$\int f(x)\,dx = F(x) + C$$

where $F(x)$ is an antiderivative of $f(x)$.

Fundamental Theorem of Calculus Pt 1

If $f$ is continuous on $[a, b]$ then the function $g$ such that

$$g(x) = \int_a^x f(t)\,dt \quad a \leq x \leq b$$

is continuous on $[a, b]$, differentiable on $(a, b)$, and $g'(x) = f(x)$.

Fundamental Theorem of Calculus Pt 2

If $f$ is continuous on $[a, b]$ then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F(x)$ is an antiderivative of $f(x)$.

Properties

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

$$\int_a^a f(x)\,dx = 0$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$\int_a^b cf(x)\,dx = c \int_a^b f(x)\,dx$$
Common Integrals

\[ \int x^n \, dx = \frac{x^{n+1}}{n+1} + C \text{ where } n \neq -1 \]
\[ \int \frac{1}{x} \, dx = \ln |x| + C \]
\[ \int e^x \, dx = e^x + C \]
\[ \int \cos (x) \, dx = \sin (x) + C \]
\[ \int \sin (x) \, dx = -\cos (x) + C \]
\[ \int \sec^2 (x) \, dx = \tan (x) + C \]
\[ \int \csc^2 (x) \, dx = -\cot (x) + C \]
\[ \int \sec (x) \tan (x) \, dx = \sec (x) + C \]
\[ \int \csc (x) \cot (x) \, dx = -\csc (x) + C \]
\[ \int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin (x) + C \]
\[ \int \frac{1}{1+x^2} \, dx = \arctan (x) + C \]
\[ \int \tan (x) \, dx = \ln |\sec (x)| + C \]
\[ \int \sec (x) \, dx = \ln |\sec (x) + \tan (x)| + C \]

Improper Integrals

Infinite Interval

\[ \int_a^\infty f(x) \, dx = \lim_{t \to \infty} \int_a^t f(x) \, dx \]
\[ \int_{-\infty}^b f(x) \, dx = \lim_{t \to \infty} \int_t^b f(x) \, dx \]
\[ \int_{-\infty}^\infty f(x) \, dx = \lim_{s \to -\infty} \int_s^b f(x) \, dx + \lim_{t \to \infty} \int_a^t f(x) \, dx \]
*assuming BOTH limits on the right converge

Discontinuous Integrand

If \( f \) is continuous on \([a, b)\) and discontinuous at \( b \) then \( \int_a^b f(x) \, dx = \lim_{t \to b^-} \int_a^t f(x) \, dx \).

If \( f \) is continuous on \((a, b]\) and discontinuous at \( a \) then \( \int_a^b f(x) \, dx = \lim_{t \to a^+} \int_t^b f(x) \, dx \).

If \( f \) has a discontinuity at \( c \) where \( a < c < b \) then

\[ \int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx \]
given that both of the integrals on the right converge.