New Quaternion-based Least-squares Method for Attitude Determination with Vector Observations

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BIOGRAPHY

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ABSTRACT

This paper presents an alternative form of attitude determination algorithm based on vector observations from GPS antenna array. This method is based on a simple iterative least-squares estimation procedure. The quaternion parameterization is adopted for the practical reason that the representation of quaternion for attitude is free from singularity problem. The proposed method estimates four elements of quaternion parameters directly from the observations without estimating the nine elements of the attitude matrix or the Direction-Cosine-Matrix (DCM). It is shown that the cost function used in our method is equivalent to that of Wahba's problem in a special case. The uncertainties in attitude solution are given in terms of a simple form of error covariance matrix.

The proposed algorithm is evaluated via simulation for the situation where the observed vectors are the estimated baselines of a GPS antenna array. The performance of the proposed algorithm is compared with other eight existing methods. They are two versions of QUaternion ESTimator (QUEST), Singular Value Decomposition (SVD) method, Fast Optimal Attitude Matrix (FOAM), Slower Optimal Matrix Algorithm (SOMA), Transformation Method (TM), Vector Observation Method (VOM), and TRIAD algorithm. Results indicate that the new algorithm accurately estimates the attitude of a moving vehicle and provides attitude uncertainties correctly.

INTRODUCTION

The problem of attitude determination is to find the rotation matrix or a set of orientation parameters which rotates the baseline vectors in the reference frame into the corresponding vectors in the body frame from a set of attitude sensor measurements [1].

The utilization of precise carrier phase measurements from GPS provides a novel approach for three-axis attitude determination [2-5]. The attitude determination methods using GPS may be classified into two types of approaches. One approach is to determine the attitude parameters directly from the differenced carrier phase measurements [6-9]. The most common scheme in this approach minimizes a cost function constituting the normalized sum weighted two-norm residuals between the measured and the known differenced carrier phase quantities as proposed by Cohen[6]

$$J(A) = \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} v_{ij}(\Delta f_i^{\,j} - b_i^T A s^j)^2 \tag{1}$$

where $s \in R^3$ is the normalized line-of-sight vector to the GPS satellite in the reference frame, $b \in R^3$ is the normalized baseline vector of the GPS antenna array in the body frame, A is the proper orthogonal attitude matrix with $A^T A = I$ and $\det(A) = 1$, T denotes a matrix transpose, subscripts i and j represent the baseline and the observed GPS satellite identifiers, respectively, and the parameter v_{ij} is a weighting factor for individual carrier phase measurement. The optimal attitude solution that minimizes this nonlinear cost function is obtained by iteration. Note that the measurement, Δf_i^j in the Cohen’s cost function is a scalar value of single differenced carrier phases measured by two GPS receivers associated with baseline i and satellite j. This type of cost function is only applicable to the specially dedicated GPS antenna array which contains more than three GPS receiver hardware modules connected to one common oscillator to eliminate the receiver-dependent clock bias error.

The other approach is to determine the attitude parameters from the estimated baseline vectors of an antenna array. The motivation behind this approach is to use the well-established result of precise relative positioning technique with GPS carrier phase measurements. Once the baseline vectors are obtained, the attitude determination becomes much simpler. In this approach the attitude determination problem may be considered as the problem of finding a proper 3-by-3 orthogonal transformation matrix which minimizes the Wahba’s loss function.

In this paper, an alternative approach is presented to determine the four elements of the quaternion directly from the baseline vector measurements instead of finding 3-by-3 transformation matrix. An instantaneous least-squares solution method is derived. This alternative method extends the Wahba’s problem into the nonlinear least-squares estimation problem with vectorized measurements and constraint equation.

The organization of this paper proceeds as follows. After a general loss function is described for GPS attitude determination from vector observations, a simple nonlinear least-squares estimation technique is introduced for estimating quaternion parameters. Constraint solutions in terms of quaternion norm are also described. It is shown that the Wahba’s cost function is equivalent to a special case of the cost function defined in this paper. Simulation results for a moving vehicle with near-optimal and nonorthogonal baseline configurations are described.

ATTITUDE DETERMINATION FROM VECTORIZED MEASUREMENTS

In this section, the problem of attitude determination from vector measurements is reviewed and a new method for attitude determination from GPS phase measurements is derived.

By using double differenced carrier phase measurements between a pair of antennas and a pair of satellites, the user can determine the relative range between the pair of antennas very precisely with a mm-level accuracy. Baseline vectors thus obtained from an array of three or more GPS antennas and multiple satellites can be used as observations for estimating the full three-axis attitude of a rigid body.

It is assumed in this paper that baseline vectors of the GPS antenna array are estimated in the WGS-84 frame (dx, dy, dz) and transformed into the local-level frame (dN, dE, dD). Either a transformation matrix or a set of orientation parameters defined in the local-level frame is estimated. The attitude determination algorithms are described on the assumption that the measured directions depend explicitly on the attitude alone not on other component such as sensor bias [17, 18]. The procedure for determining attitude from GPS measurements is summarized in Figure 1.

Wahba’s Problem

The problem of attitude determination from vector measurements can be considered as the problem of determining an orthogonal transformation matrix A in the following relationship:

$$\underline{r}_i = A \cdot \underline{l}_i + \underline{e}_i, \quad \text{for } i=1,\ldots,n \tag{2}$$

where $\underline{l}_i$ denotes the vector representation of the direction to some observed object in the local-level frame, $\underline{r}_i$ is the previously defined vector representation of the corresponding observation in the vehicle body frame, $\underline{e}_i$ represents an error vector, and n is the number of vector observations.

Wahba posed the problem of finding the proper orthogonal matrix A that minimizes the loss function defined as [10]:

$$J(A) \equiv \frac{1}{2}\sum_{i=1}^{n} v_i \left|\underline{r}_i - A \cdot \underline{l}_i\right|^2 \tag{3}$$

where v_i represents a weighting factor. By simple

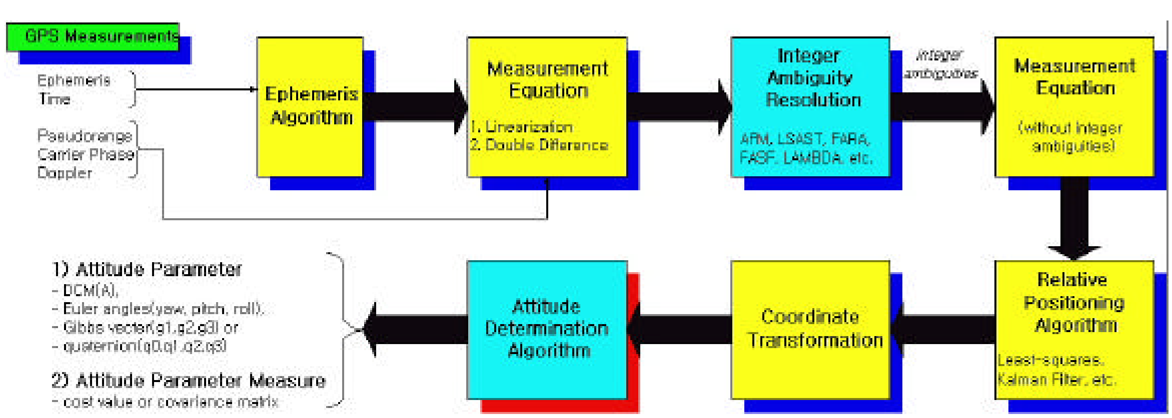

Figure 1. Procedure for attitude determination with GPS measurements

matrix manipulations, this Wahba's problem is equivalent to the problem of finding the proper orthogonal matrix A that maximizes the trace of the matrix product AB^T where the matrix B is given by

$$B = \sum_{i=1}^{n} v_i \underline{r}_i \underline{l}_i^T \quad (4).$$

Davenport suggested a solution to the Wahba's quadratic loss function by utilizing quaternion method [11]. Many practical attitude determination algorithms have been developed based on this method. This method by Davenport, called the q-method, directly led to an eigenvalue equation for quaternion by Keat [12] and the QUaternion ESTimator (QUEST) by Shuster [13]. The QUEST was derived by using either Gibbs vector or quaternion. Markley suggested an algorithm based on the Singular Value Decomposition (SVD) method focusing on theoretical analysis and robust computing [14]. The algorithm requires extra computation for the singular value decomposition. He also presented two improved SVD solutions to the Wahba's problem, known as Fast Optimal Attitude Matrix (FOAM) and Slower Optimal Matrix Algorithm (SOMA), without performing the singular value decomposition [15]. A more complete survey of other attitude representations is given in Reference 16.

Quaternion-based Least-squares Method

Given a set of reference vectors and a corresponding set of directional measurements between a body axis and these reference vectors, the problem of three-axis attitude determination can be formulated as a linear least squares problem with norm constraint on the solution [19]. The attitude determination algorithm based on a simple iterative least-squares can easily be derived for three element parameters of Euler angle [3]. The transformation matrix derived from Euler angles can take any of the 12 possible forms depend on the sequence of rotation angles. Since the attitude matrix derived from Euler angles is not inherently nonsingular, special procedures are needed to deal with singularity of 180 degree rotations. The problem of singularity can be solved by using a quaternion parameterization of the rotation matrix. In this section, a complete solution algorithm for estimating quaternion parameters based on least-squares with quaternion constraints is developed. The solution of the proposed algorithm is equivalent to the solution of the Wahba's problem in the case explained later.

The four elements of quaternion parameters used in this paper are defined as [20]:

$$\underline{q} = (q_0, q_1, q_2, q_3)^T = q_0 + q_1\mathrm{i} + q_2\mathrm{j} + q_3\mathrm{k} = (q_0, \mathrm{q}^T)^T \quad (5),$$

where q_0 is the amount of rotation about a vector defined by q_1, q_2 and q_3 in space. The transformation matrix derived from these quaternion parameters can be uniquely defined as:

$$A^T = \begin{bmatrix} q_0^2+q_1^2-q_2^2-q_3^2 & 2(q_3q_0+q_1q_2) & 2(q_1q_3-q_0q_2) \\ 2(q_1q_2-q_3q_0) & q_0^2-q_1^2+q_2^2-q_3^2 & 2(q_0q_1+q_3q_2) \\ 2(q_0q_2+q_1q_3) & 2(q_2q_3+q_0q_1) & q_0^2-q_1^2-q_2^2+q_3^2 \end{bmatrix} \quad (6).$$

Substitution of the transformation matrix A of Eq. (6) into Eq. (2) for any i and with some manipulation yields another nonlinear observation equation in terms of quaternion parameters as:

$$\begin{aligned} l_{N_i} &= (q_0^2+q_1^2-q_2^2-q_3^2)r_x^{b_i} + 2(q_3q_0+q_1q_2)r_y^{b_i} + 2(q_1q_3-q_0q_2)r_z^{b_i} \\ l_{E_i} &= 2(q_1q_2-q_3q_0)r_x^{b_i} + (q_0^2-q_1^2+q_2^2-q_3^2)r_y^{b_i} + 2(q_0q_1+q_3q_2)r_z^{b_i} \\ l_{D_i} &= 2(q_0q_2+q_1q_3)r_x^{b_i} + 2(q_2q_3+q_0q_1)r_y^{b_i} + (q_0^2-q_1^2-q_2^2+q_3^2)r_z^{b_i} \end{aligned} \quad (7)$$

which can be written in a vector form as:

$$\underline{l}_i = h_i(X_{\underline{q}}) + \underline{w}_i \quad (8)$$

where $X_{\underline{q}}$ is the vector of unknowns $(q_0, q_1, q_2, q_3)^T$. Linearization of Eq. (8) about nominal values yields

$$d\,\underline{l}_{i\underline{q}} = H_{i\underline{q}} \cdot dX_{\underline{q}} + \underline{v}_i \quad (9)$$

where $d\,\underline{l}_{i\underline{q}}$ is the vector of corrections of the observations, $dX_{\underline{q}}$ represents the corrected unknown quaternion vector and $H_{i\underline{q}}$ is the matrix of partial derivatives which has the general form as:

$$H_{i\underline{q}} = \frac{\partial h_i(X_{\underline{q}})}{\partial X_{\underline{q}}}\Big|_{X_{\underline{q}}=X_{\underline{q}}^*} = \begin{bmatrix} \frac{\partial h_N}{\partial q_0} & \frac{\partial h_N}{\partial q_1} & \frac{\partial h_N}{\partial q_2} & \frac{\partial h_N}{\partial q_3} \\ \frac{\partial h_E}{\partial q_0} & \frac{\partial h_E}{\partial q_1} & \frac{\partial h_E}{\partial q_2} & \frac{\partial h_E}{\partial q_3} \\ \frac{\partial h_D}{\partial q_0} & \frac{\partial h_D}{\partial q_1} & \frac{\partial h_D}{\partial q_2} & \frac{\partial h_D}{\partial q_3} \end{bmatrix} \quad (10)$$

where

$$\frac{\partial h_N}{\partial q_0} = 2(q_0r_x^{b_i} - q_3r_y^{b_i} + q_2r_z^{b_i}),\ \frac{\partial h_N}{\partial q_1} = 2(q_1r_x^{b_i} + q_2r_y^{b_i} + q_3r_z^{b_i}),$$

$$\frac{\partial h_N}{\partial q_2} = 2(-q_2r_x^{b_i} + q_1r_y^{b_i} + q_0r_z^{b_i}),\ \frac{\partial h_N}{\partial q_3} = 2(-q_3r_x^{b_i} - q_0r_y^{b_i} + q_1r_z^{b_i}),$$

$$\frac{\partial h_E}{\partial q_0} = 2(q_3r_x^{b_i} + q_0r_y^{b_i} - q_1r_z^{b_i}),\ \frac{\partial h_E}{\partial q_1} = 2(q_2r_x^{b_i} - q_1r_y^{b_i} - q_0r_z^{b_i}),$$

$$\frac{\partial h_E}{\partial q_2} = 2(q_1r_x^{b_i} + q_2r_y^{b_i} + q_3r_z^{b_i}),\ \frac{\partial h_E}{\partial q_3} = 2(q_0r_x^{b_i} - q_3r_y^{b_i} + q_2r_z^{b_i}),$$

$$\frac{\partial h_D}{\partial q_0} = 2(-q_2r_x^{b_i} + q_1r_y^{b_i} + q_0r_z^{b_i}),\ \frac{\partial h_D}{\partial q_1} = 2(q_3r_x^{b_i} + q_0r_y^{b_i} - q_1r_z^{b_i}),$$

$$\frac{\partial h_D}{\partial q_2} = 2(-q_0r_x^{b_i} + q_3r_y^{b_i} - q_2r_z^{b_i}),\ \frac{\partial h_D}{\partial q_3} = 2(q_1r_x^{b_i} + q_2r_y^{b_i} + q_3r_z^{b_i}).$$

It is shown in the above expressions for the partial derivatives that only four different elements need to be computed; i.e., the elements in the second and the third rows of the matrix can readily be obtained by using the elements in the first row of the matrix. Thus, the partial derivative matrix can simply be written as

$$H_{i\underline{q}} = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ -a_4 & -a_3 & a_2 & a_1 \\ a_3 & -a_4 & -a_1 & a_2 \end{bmatrix} \quad (11).$$

The attitude solution can be obtained by expanding Eq. (9) to a matrix form and applying a simple iterative least-squares with the following correction equation:

$$d\hat{X}_{\underline{q}} = (H_{\underline{q}}{}^T Q^{-1} H_{\underline{q}})^{-1} H_{\underline{q}}{}^T Q^{-1} \cdot dL_{\underline{q}} \quad (12),$$

where

$$dL_{\underline{q}} = \begin{bmatrix} d\underline{l}_{1\underline{q}}^T & d\underline{l}_{2\underline{q}}^T & \cdots & d\underline{l}_{m\underline{q}}^T \end{bmatrix}^T,$$
$$H_{\underline{q}} = \begin{bmatrix} H_{1\underline{q}}^T & H_{2\underline{q}}^T & \cdots & H_{m\underline{q}}^T \end{bmatrix}^T,$$

and Q represents the noise covariance matrix.

Least-squares with Quaternion Constraint

The attitude matrix obtained from the solution in Eq. (12) may not be proper orthogonal because of the noise contained in the measurements. This problem can be corrected by applying the orthogonal constraint of the quaternion parameters which has a form of nonlinear equation as:

$$q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1 \quad (13).$$

The solution to this constraint problem can be obtained by first linearizing Eq. (13) about nominal values as:

$$dl_c = H_c \cdot dX_{\underline{q}} \quad (14)$$

where

$$dl_c = 1 - (q_0^2 + q_1^2 + q_2^2 + q_3^2),$$
$$H_c = \begin{bmatrix} 2q_0^2 & 2q_1^2 & 2q_2^2 & 2q_3^2 \end{bmatrix}.$$

The correction vector for the iterative least-squares solution and its covariance matrix are then given by [21]

$$d\tilde{X}_{\underline{q}} = (H^T H)^{-1}\left[H^T dL + H_c^T\{H_c(H^T H)^{-1}H_c^T\}^{-1}\{1 - H_c(H^T H)^{-1}HdL\}\right] \quad (15)$$

$$Cov(d\tilde{X}_{\underline{q}}) = s^2\left[(H^T H)^{-1} - (H^T H)^{-1}H_c^{-1}\{H_c(H^T H)^{-1}H_c^{-1}\}^{-1}H_c(H^T H)^{-1}\right] \quad (16).$$

The covariance of the attitude estimate in Eq. (16) is a statistical measure of the estimation errors arising from errors in the reference and observation vectors. The relationship between constraint least-squares and unconstrained least-squares is given by

$$d\tilde{X}_{\underline{q}} = d\hat{X}_{\underline{q}} - (H^T H)^{-1} H_c^T \{H_c (H^T H)^{-1} H_c\}^{-1} (H_c d\hat{X}_{\underline{q}} - 1) \quad (17).$$

The solution of constrained least-squares approach described in Eqs. (15) and (17) provides more accurate results than the unconstrained least-squares in Eq. (12) in general, although the difference is often negligible. The constrained solution takes considerably more computations than the non-constained solution. This additional computational time can be reduced by using a simple ad hoc approach. A common ad hoc approach, especially in the field of attitude determination, where one often has to solve a linear system with normalization constraint, is to solve the unconstrained linear system and then to normalize the solution to unity [19]:

$$\tilde{X}_{\underline{q}} = \hat{X}_{\underline{q}} \big/ \left|\hat{X}_{\underline{q}}\right| \quad (18).$$

Cost Function of Least-squares

This section shows that the Wahba's problem is a special case of our nonlinear least-squares problem. For a multi-baseline situation, the cost function of nolinear least-squares in terms of four elements of quaternion parameters in Eq. (12) is given in a matrix form as:

$$J(X) = \frac{1}{2}\left(L - h(X)\right)^T W \left(L - h(X)\right) \quad (18)$$

where

$$L = \begin{bmatrix} \underline{l}_1^T & \underline{l}_2^T & \cdots & \underline{l}_m^T \end{bmatrix}^T,$$
$$h(X) = \begin{bmatrix} h_1^T(X) & h_2^T(X) & \cdots & h_m^T(X) \end{bmatrix}^T,$$
$$X = (y, q, f)^T \text{ or } (q_0, q_1, q_2, q_4)^T,$$
$$W = diag\{W_1, W_2, \cdots, W_m\}$$

and W_i denotes a weighting matrix corresponding to measurement noise statistics of the i-th baseline.

This cost function can be written in another matrix form as:

$$J(X) = \frac{1}{2}\left(L - C_X \cdot R\right)^T W \left(L - C_X \cdot R\right) \quad (19)$$

where

$$R = \begin{bmatrix} \underline{r}_1^T & \underline{r}_2^T & \cdots & \underline{r}_m^T \end{bmatrix}^T,$$
$$C_X = diag\{A_X^T, A_X^T, \cdots, A_X^T\},$$

and A_X^T is an attitude transformation matrix expressed in terms of unknowns (q_0, q_1, q_2, q_4) as in Eq. (6), respectively. If we assume that the weighting matrix W has the form of a diagonal matrix as $W=diag\{W_1, W_2, \ldots, W_m\}$ where W_i is a weighting matrix for individual baseline, then the matrix form of cost function in Eq. (19) can be rewritten in terms of the individual attitude matrix, A_X^T as:

$$J(X) = \frac{1}{2}\sum_{i=1}^{m}\left(\underline{l}_i - A_X^T \cdot \underline{r}_i\right)^T W_i \left(\underline{l}_i - A_X^T \cdot \underline{r}_i\right) \quad (20).$$

If the weighting matrix for each baseline measurement is a scalar times the identity matrix, i.e., $W_i = 1/s_i^2 \cdot I$, where I denotes the 3-by-3 identity matrix, the resulting cost function of Eq. (20) reduces to

$$J(X) = \frac{1}{2}\sum_{i=1}^{m}\frac{1}{s_i^2}\left|\underline{l}_i - A_X^T \cdot \underline{r}_i\right|^2 \quad (21).$$

As the result, the cost function of nonlinear least-squares in attitude determination problem in Eq. (21) is equivalent to that of Wahba's problem in Eq. (2). The Wahba's cost function is a special case of nonlinear least-squares problem that a vector measurement has one same weighting factor, v_i, for the i-th baseline. If the baselines do not form an orthonormal basis, then the attitude solution is suboptimal.

SIMULATION

Three forms of our new algorithms for estimating the quaternion parameters; namely, the unconstrained least-squares (QULS), the constrained least-squares (QCLS) and the constrained least-squares of ad hoc approach (QACLS) proposed by this paper and a nonlinear least-squares algorithm for Euler angles (ELS) were coded in double precision MATLAB and executed on a Pentium III 650 MHz computer with Window Millennium Edition Operating System. In order to compare their performance with other existing algorithms, Eight attitude determination algorithms were also simulated. They are TM, VOM, TRIAD and QUEST-based methods such as two versions of QUEST, SVD, FOAM and SOMA. Brief reviews, computational steps and several simulation results of these attitude determination algorithms are summarized in Reference 22.

Four test cases were simulated as shown in Table 1. Each test cases were specified by a set of unity baseline vectors, $\underline{r}_i$ and standard deviations of carrier phase measurements, s_Φ. The VOM is not applicable to the situation when $\underline{r}_1$ is not placed on the longitudinal axis. Also it can accommodate only two observations; i.e., it does not use the additional redundant vector measurements. The third measurement vector, $\underline{r}_3 = [0,0,-1]^T$ was used only in the TM in our simulation. Other algorithms use only two measurement vectors, $\underline{r}_1$ and $\underline{r}_2$. The time history of the true Euler angles in the simulation are shown in Figure 2. The performance of each algorithms are compared in terms of the root-mean-squared Euler angle errors as shown in Table 2. The attitude error from TRIAD is relatively large because its accuracy depends on the first choice of reference-observation vector pair. The QUEST-based methods such as two versions of QUEST, SVD, FOAM and SOMA which were derived from Davenport's q-method have the same error characteristics as described in Reference 14. The iteration process in the QUEST for estimating quaternion caused large computation time to converge to its global minimum in computing l from characteristic equation, $p(l)$. QULS, QCLS and QACLS have the slightly different error characteristics, but not significantly, mainly due to their different approaches in dealing with the constraint condition. The QULS is preferable if small deviations from orthogonality can be tolerated. The output of ELS shows large errors, as shown in Figure 3, when the yaw attitude angle approaches 180 degrees because of the singularity associated with the Euler expressions. In Figure 4 depict the performances of the QUEST-based algorithms and the proposed algorithms. It is shown in the figure that the proposed algorithms have good performance comparable to those of the most efficient existing algorithms.

CONCLUSIONS

In this paper, a new alternative form of attitude estimation algorithm was proposed for the situation when the observed vectors are the estimated baselines of a GPS antenna array. The algorithm was first derived for an unconstrained nonlinear least-squares solution of the four elements of quaternion parameters. Two variants of the algorithm with a constraint on quaternion norm were also given. The second variant of the solution based on an ad hoc constraint approach reduces computation significantly with little loss in accuracy. It was shown that the cost function of the proposed least-squares based attitude estimation algorithms is equivalent to that of Wahba's problem in a special case. The performance of the proposed algorithm was compared with existing attitude determination algorithms via simulation.

Table 1. Simulation Test Cases

	Body Frame Vector(m)		
Case	$\underline{r}_1^T$	$\underline{r}_2^T$	s_Φ (mm)
1	[1, 0, 0]	[0, 1, 0]	2
2	[1, 0, 0]	[0, 1, 0]	20
3	[1, 0, 0]	$[1/\sqrt{2}, 1/\sqrt{2}, 0]$	2
4	[1, 0, 0]	$[1/\sqrt{2}, 1/\sqrt{2}, 0]$	20

Table 2. Comparison of RMS Errors

		Case 1	Case 2	Case 3	Case 4
VOM	yaw	0.101	0.901	0.088	0.979
	pitch	0.245	2.670	0.191	2.660
	roll	0.339	2.175	N/A	N/A
TM	yaw	0.101	0.901	0.089	0.979
	pitch	0.262	2.922	0.217	2.867
	roll	1.369	2.198	0.308	3.700
TRIAD	yaw	1.231	1.433	1.254	1.702
	pitch	11.305	11.813	11.240	11.988
	roll	6.477	6.048	6.505	6.755
QUEST based methods	yaw	0.081	0.769	0.069	0.878
	pitch	0.245	2.681	0.191	2.634
	roll	0.285	2.075	0.275	3.448
QACLS	yaw	0.081	0.812	0.068	0.877
	pitch	0.246	2.445	0.190	2.634
	roll	0.285	2.870	0.274	3.447

Unit : Deg

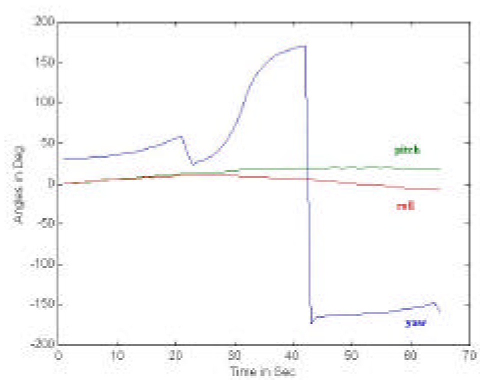

Figure 2. True Euler angles

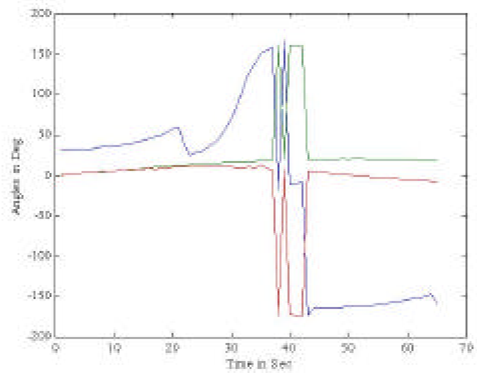

Figure 3. Euler Angle Outputs of ELS

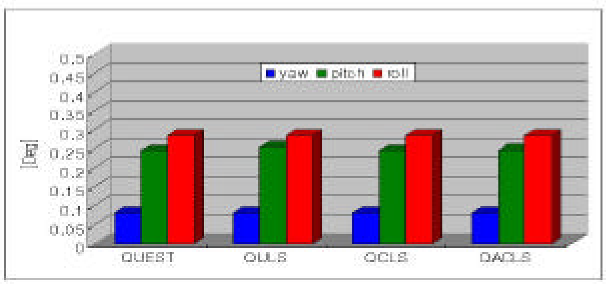

Figure 4. Comparison of Algorithms

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