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ENCH 501: Transport Processes Quiz #7 **December 5, 2006**

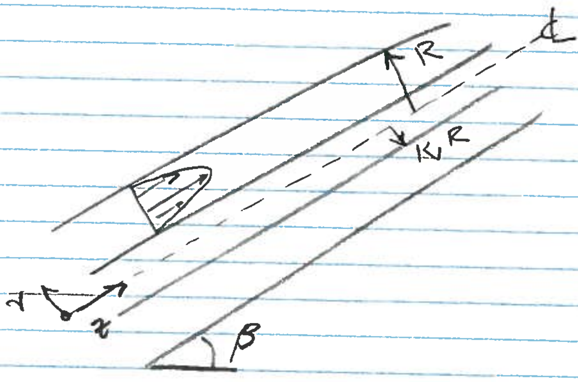
Time Allowed: 45 mins. **Name:**

Two concentric tubes are inclined at angle β to the horizon. The inner radius of the outer tube is **R** and the outer radius of the inner tube is κ**R**. A fluid flows at a steady rate through the annular space in the upwards direction. (Fluid density = ρ and viscosity = μ).

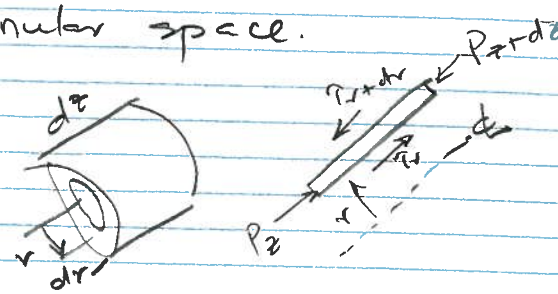
a) Derive an expression for the velocity distribution in the annular space. Show all your steps.

b) What is the relationship between the maximum and average velocities if $\kappa = 0.4$?

ENCH 501 Quiz 7 Solution Dec. 5, 2006



Consider a differential element – a ring in the annular space.



Since the flow is steady and pipe radii are constant, there is no change in momentum. The force balance is given by

$$(2\pi r\,dz\,\tau_r)\Big|_r - (2\pi r\,dz\,\tau_r)\Big|_{r+dr} + P_z(2\pi r\,dr) - P_{z+dz}(2\pi r\,dr) - (2\pi r\,dr\,dz)\rho g\sin\beta = 0$$

or

$$-2\pi\,dz\,\frac{d(r\tau_r)}{dr}\,dr - \frac{dP}{dz}\,dz\,(2\pi r\,dr) - \rho g\sin\beta\,(2\pi r\,dr\,dz) = 0$$

or

$$\frac{d(r\tau_r)}{dr} + r\frac{dP}{dz} + r\rho g\sin\beta = 0$$

when $\tau_r = -\mu\frac{du}{dr}$ is substituted

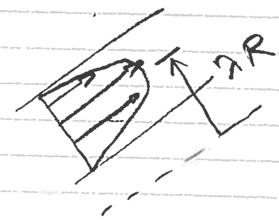
$$\frac{1}{r}\frac{d}{dr}\left(r\frac{\partial u}{\partial r}\right) = \frac{1}{\mu}\left\{\frac{dP}{dz} + \rho g\sin\beta\right\} \qquad (1)$$

Define, $P = \frac{dP}{dz} + \rho g\sin\beta = \text{constant}$

Equation (1) is subject to the boundary conditions

$$r = \kappa R \quad u = 0 \quad \text{and} \quad r = R, \quad u = 0 \qquad (2)$$

Integrate (1)



$$r\frac{du}{dr} = \frac{\Gamma}{\mu}\frac{r^2}{2} + C_1$$

If the maximum velocity is at $r = \lambda R$, where $\frac{du}{dr} = 0$, then $C_1 = -\frac{\Gamma}{2\mu}(\lambda R)^2$, i.e. an unknown λ has been substituted for another unknown C_1

$$r\frac{du}{dr} = \frac{\Gamma}{2\mu}\left(r^2 - \lambda^2 R^2\right) \qquad (3)$$

Integrate

$$u = \frac{\Gamma}{2\mu}\left(\frac{r^2}{2} - \lambda^2 R^2 \ln r\right) + C_2 \qquad (4)$$

use the boundary conditions

$$0 = \frac{\Gamma}{2\mu}\left(\frac{R^2}{2} - \lambda^2 R^2 \ln R\right) + C_2$$

$$0 = \frac{\Gamma}{2\mu}\left(\frac{\kappa^2 R^2}{2} - \lambda^2 R^2 \ln \kappa R\right) + C_2$$

Subtract

$$0 = \frac{\Gamma}{2\mu}\left\{\frac{R^2}{2}\left(1-\kappa^2\right) - \lambda^2 R^2 \ln\frac{R}{\kappa R}\right\}$$

$$\therefore \quad \lambda^2 = \frac{1-\kappa_1^2}{2\ln(1/\kappa_1)} \qquad (5)$$

and

$$C_2 = -\frac{\Gamma R^2}{2\mu}\left\{\frac{1}{2} - \frac{(1-\kappa_1^2)}{2\ln(1/\kappa_1)}\ln R\right\} \qquad (6)$$

Subst. into (4)

$$u = \frac{\Gamma R^2}{2\mu}\left\{\frac{1}{2}\frac{r^2}{R^2} - \lambda^2 \ln r\right\} - \frac{\Gamma R^2}{2\mu}\left\{\frac{1}{2} - \lambda^2 \ln R\right\}$$

$$u = \frac{\Gamma R^2}{2\mu}\left\{\frac{1}{2}\left(\frac{r^2}{R^2}-1\right) - \lambda^2 \ln\frac{r}{R}\right\} \qquad (7)$$

overall

(a)
$$u = \frac{\left[\frac{dP}{dz} + \rho g \sin\beta\right]R^2}{4\mu}\left\{\frac{r^2}{R^2} - 1 - \frac{1-\kappa_1^2}{\ln(1/\kappa_1)}\ln\frac{r}{R}\right\}$$

→

(b) The velocity is maximum at $r = \lambda R$ or $\frac{r}{R} = \lambda$

$$u_{max} = \frac{\Gamma R^2}{4\mu}\left\{\lambda^2 - 1 - 2\lambda^2 \ln\lambda\right\} \qquad (8)$$

The average velocity

$$\bar{u} = \frac{\int_{\kappa_1 R}^{R} 2\pi r\, dr\, u}{\int_{\kappa_1 R}^{R} 2\pi r\, dr} = \frac{\int_{\kappa_1 R}^{R} r u\, dr}{\int_{\kappa_1 R}^{R} r\, dr}$$

$$\bar{w} = \frac{\int_{\kappa R}^{R} r\left(\frac{\Gamma R^2}{4\mu}\right)\left(\frac{r^2}{R^2} - 1 - 2\lambda^2 \ln r/R\right) dr}{\int_{\kappa R}^{R} r\, dr}$$

$$= \frac{\frac{\Gamma R^2}{4\mu}\int_{\kappa R}^{R}\left(\frac{r^3}{R^2} - r - 2\lambda^2 r \ln r/R\right) dr}{\frac{1}{2}R^2(1-\kappa^2)}$$

$$= \frac{\frac{\Gamma R^2}{\cancel{4}\mu}\cdot \cancel{R^2}\int_{\kappa}^{1}\left(\frac{r^3}{R^3} - \frac{r}{R} - 2\lambda^2 \frac{r}{R}\ln\frac{r}{R}\right) d(r/R)}{\cancel{\tfrac{1}{2}}\,\cancel{R^2}(1-\kappa^2)}$$

$$= \frac{\Gamma R^2}{2\mu(1-\kappa^2)}\int_{\kappa}^{1}\left(\eta^3 - \eta - 2\lambda^2\eta\ln\eta\right) d\eta$$

$$= \frac{\Gamma R^2}{2\mu(1-\kappa^2)}\left[\frac{\eta^4}{4} - \frac{1}{2}\eta^2 - 2\lambda^2\left(\frac{\eta^2}{2}\left(\ln\eta - \frac{1}{2}\right)\right)\right]_{\kappa}^{1}$$

$$= \frac{\Gamma R^2}{2\mu(1-\kappa^2)}\left[\left(\frac{1}{4} - \frac{1}{2} - 2\lambda^2\left\{\frac{1}{2}\left(\ln 1 - \frac{1}{2}\right)\right\}\right) - \left(\frac{\kappa^4}{4} - \frac{\kappa^2}{2} - 2\lambda^2\left\{\frac{\kappa^2}{2}\left(\ln\kappa - \frac{1}{2}\right)\right\}\right)\right]$$

$$= \frac{PR^2}{2\mu(1-\kappa^2)}\left[\left(-\frac{1}{4}+\frac{2\lambda^2}{4}\right)-\left(\frac{\kappa^4}{4}-\frac{\kappa^2}{2}-\frac{2\lambda^2\kappa^2}{2}\ln\kappa+\frac{2\lambda^2\kappa^2}{4}\right)\right]$$

substitute here

Given that $2\lambda^2 = \dfrac{1-\kappa^2}{\ln(1/\kappa)}$

$$= \frac{PR^2}{2\mu(1-\kappa^2)}\left[-\frac{1}{4}+\frac{2\lambda^2(1-\kappa^2)}{4}-\frac{\kappa^4}{4}+\frac{\kappa^2}{2}-\frac{1-\kappa^2}{\cancel{\ln(1/\kappa)}}\,\frac{\kappa^2}{2}\,\cancel{\ln(1/\kappa)}\right]$$

* (Note: $-\ln\kappa = \ln\left(\frac{1}{\kappa}\right)$)

$$= \frac{PR^2}{2\mu(1-\kappa^2)}\left[-\frac{1}{4}+\frac{2\lambda^2(1-\kappa^2)}{4}-\frac{\kappa^4}{4}+\frac{\kappa^4}{2}\right]$$

$$= \frac{PR^2}{8\mu(1-\kappa^2)}\left[(1-\kappa^4)-\frac{(1-\kappa^2)(1-\kappa^2)}{\ln(1/\kappa)}\right]$$

$$\bar{u} = -\frac{PR^2}{8\mu}\left[\frac{(1-\kappa^4)}{(1-\kappa^2)}-\frac{(1-\kappa^2)}{\ln(1/\kappa)}\right] \quad (9)$$

From equation (8)

$$U_{max} = -\frac{\Gamma R^2}{4\mu}\left\{1 - \frac{(1-\kappa^2)}{2\ln(1/\kappa)} + \frac{1-\kappa^2}{\ln(1/\kappa)}\ln\left[\frac{1-\kappa^2}{2\ln(1/\kappa)}\right]^{\frac{1}{2}}\right\}$$

The ratio is :

$$\frac{\bar{u}}{U_{max}} = \frac{1}{2}\left\{\frac{\dfrac{1-\kappa^2}{\ln(1/\kappa)} - \dfrac{1-\kappa^4}{1-\kappa^2}}{\dfrac{1-\kappa^2}{2\ln\frac{1}{\kappa}} - 1 - \dfrac{1-\kappa^2}{\ln(1/\kappa)}\ln\left(\dfrac{1-\kappa^2}{2\ln(1/\kappa)}\right)^{\frac{1}{2}}}\right\} \quad (10)$$

$$\frac{(1-\kappa^2)}{2\ln(1/\kappa)} = \frac{1-0.4^2}{2\ln(2.5)} = 0.4584$$

$$\frac{\bar{u}}{U_{max}} = \frac{0.4584 - \dfrac{(1-0.4^4)}{2(1-0.4^2)}}{0.4584 - 1 - 0.4584\ln 0.4584}$$

$$= 0.6608$$

$$\bar{u} = 0.6608\, U_{max}$$

$\longrightarrow$