GJ

The University of Calgary
Department of Chemical & Petroleum Engineering

ENCH 501: Transport Phenomena Quiz #6 November 22, 2011

Time Allowed: 45 mins. Name:

A current is passed through a long metal wire and heat is assumed generated uniformly through the wire at a constant rate of g^+ per unit volume. The wire was initially at the ambient temperature $\mathbf{T_\infty}$ and the heat transfer coefficient around the wire is **h**.

Derive an expression for the temperature **T(r,t)** in the wire. Use the Integral Method and show all important steps. State your assumptions.

(Examples of this are wires in electric blankets, tungsten filaments in light bulbs and wires in various sensors and detectors used in the chemical and petroleum industries.)

Hints: Define a new temperature scale, $\theta = T(r,t) - T_\infty = Y(r)\Gamma(t)$. The Biot number, Bi, is hR/k. Y(r) is the steady state solution. $\Gamma(0) = 0$ and $\Gamma(\infty) = 1$.

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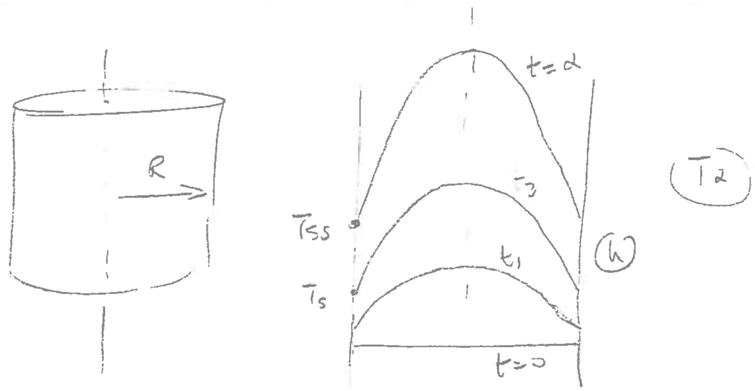

Let T_{ss} = steady state surface temp. and $T_s(t)$ is transient surface temperature

Energy Balance eq.

$$\underset{gen}{g^+(\pi R^2 L)} = \underset{output}{q_R(2\pi R L)} + \frac{d}{dt}\underset{accum}{\left[\int_0^R (T-T_\infty)\rho C_p\, 2\pi L r\, dr\right]}$$

or
$$g^+ R^2 = -k\left.\frac{dT}{dr}\right|_R (2R) + \frac{d}{dt}\left[2\rho C_p R^2 \int_0^1 (T-T_\infty)\frac{r}{R}\frac{dr}{R}\right]$$

or
$$\frac{g^+}{2\rho C_p} = -\frac{k}{\rho C_p R}\left.\frac{dT}{dr}\right|_R + \frac{d}{dt}\left[\int_0^1 (T-T_\infty)\frac{r}{R}\frac{dr}{R}\right] \quad (1)$$

Assume $T(r,t) - T_\infty = \theta(r,t) = Y(r)\Gamma(t)$ (2)

where $Y(r)$ is the steady state solution $\Rightarrow$ $\Gamma(\infty) = 1$ and $\Gamma(0) = 0$

Conditions at Steady State: $r=0$ $\frac{\partial T}{\partial r} = 0$ (symmetry) (3)

$r=R$ $-k\left.\frac{dT}{dr}\right|_R (2\pi R L) = g^+(\pi R^2)L$ no accum

and $r = R$, $T = T_{ss}$ ($> T_\infty$ when h finite)

i.e. $\theta(r,\infty) = Y(r) = a + br + cr^2$ (4)

$$\frac{dY}{dr} = b + 2cr$$

$r=0$ $\frac{dY}{dr} = 0 \Rightarrow b = 0$

$r=R$ $\frac{dY}{dr} = 2cR = -\frac{g^+R}{2k} \Rightarrow c = -\frac{g^+}{4k}$

at $r = R$ $\quad T_{ss} - T_\alpha = a - \frac{g^+ R^2}{4k}$

$$\therefore \quad Y(r) = \theta(r, \alpha) = (T_{ss} - T_\alpha) + \frac{g^+ R^2}{4k} - \frac{g^+ r^2}{4k}$$

Using the boundary condition (at steady state)

$$-k \left.\frac{dT}{dr}\right|_R (2\pi R L) = h(2\pi R L)(T_{ss} - T_\alpha) = g^+ (\pi R^2 L)$$

$$T_{ss} - T_\alpha = \frac{g^+ R}{2h}$$

$$\therefore \quad Y(r) = \frac{g^+ R}{2h} + \frac{g^+ R^2}{4k}\left(1 - \frac{r^2}{R^2}\right)$$

$$\therefore \quad T(r,t) - T_\alpha = \frac{g^+ R^2}{4k}\left\{1 - \frac{r^2}{R^2} + \frac{4k}{2hR}\right\}\Gamma(t) \qquad (5)$$

$$\text{or} \quad T(r,t) - T_\alpha = \beta\left(1 - \frac{r^2}{R^2} + \frac{2}{Bi}\right)\Gamma(t)$$

where $\beta = \frac{g^+ R^2}{4k}$ and $Bi = \frac{hR}{k}$

If $\eta = r/R$, the integral energy equation is

$$\frac{g^+}{2\rho c_p} = -\frac{\alpha}{R^2}\left.\frac{dT}{d\eta}\right|_{\eta=1} + \frac{d}{dt}\left[\int_0^1 \beta\left(1 - \eta^2 + \frac{2}{Bi}\right)\Gamma\, \eta\, d\eta\right]$$

$$\frac{g^+}{2\rho c_p} = +\frac{\alpha}{R^2} 2\beta\Gamma + \frac{d}{dt}\left[\beta\Gamma\int_0^1 \left(1 - \eta^2 + \frac{2}{Bi}\right)\eta\, d\eta\right]$$

$$\frac{g^+}{2\rho C_p}\cdot\frac{1}{\beta} = \frac{\cancel{g^+}}{\cancel{2}\rho C_p}\frac{\overset{2}{\cancel{4}}k}{\cancel{g^+}R^2} = \frac{2\alpha}{R^2} = \frac{2\alpha}{R^2}\Gamma + \frac{d}{dt}\left[\Gamma\left(\frac{1}{4}+\frac{1}{Bi}\right)\right]$$

$$\left(\frac{1}{4}+\frac{1}{Bi}\right)\frac{d\Gamma}{dt} + \frac{2\alpha}{R^2}(\Gamma-1) = 0 \qquad (6)$$

subject to $t=0$, $\Gamma=0$

Solve

$$\Gamma = 1 - \exp\left[-\frac{2\alpha}{R^2}\left(\frac{4Bi}{4+Bi}\right)t\right] \qquad (7)$$

Subst. (7) in (5)

$$T(r,t) - T_\alpha = \frac{g^+R^2}{4k}\left\{1 - \frac{r^2}{R^2} + \frac{2}{Bi}\right\}\left(1 - \exp\left[-\frac{2\alpha}{R^2}\cdot\frac{4Bi}{4+Bi}\cdot t\right]\right) \qquad (8)$$

$\longrightarrow$