ENCH 501 F2017 **Quiz #5** Name:______________________________

November 14, 2017 Time Allowed: 45 minutes aJ

People who work or play outside in cold environments appreciate hand and feet warmers. These are units within which heat is generated electrically by a battery, by chemical reactions or by the release of latent heat from super-saturated solutions such as sodium acetate. A unit that operates by chemical oxidation consists of iron powder (called filings), ground charcoal, salt, sawdust and vermiculite in a pouch that is freely permeable to air once removed from its packaging seal. Oxygen from air reacts with the iron to form iron oxides or rust. The reaction is exothermic and the heat released is both stored in the unit and released to the outside. Salt catalyzes the reaction, carbon disperses the heat, vermiculite acts as storage and sawdust absorbs moisture. As the iron is progressively oxidized, the reaction rate slows and heat generation declines.

Consider a large rectangular pouch containing all the above components that are well mixed that the combination is treated as a "homogeneous solid". One side of the pouch is placed flat on an insulating surface and the packaging removed to suddenly expose the other side to ambient air at -20°C. The solid layer is 12 mm thick. The oxidation reaction starts immediately, and it is assumed that the rate of heat generation per unit volume is constant and uniform throughout the unit for the period of interest. The convective heat transfer coefficient external to the exposed surface of the pouch is 7.5 W/m^2 K. The density, the thermal conductivity of the mixture (solid) and the heat capacity are respectively 160 kg/m^3, 0.09 W/m K and 0.94 kJ/kg K. The pouch was originally at the ambient temperature at the instant the packaging was removed. The temperature of the exposed surface was measured to be 10°C after 5 minutes from the start. ? -15°C

a) Obtain an expression for the temperature across the solid layer as a function of position and time, T(x, t). Use the **integral method** and show your steps.
b) Estimate the rate of heat generation per unit volume in the solid.
c) Estimate the steady state temperatures at the insulated wall and at the exposed surface.

Solution for Quiz #5

a) This problem is similar to the unsteady heat transfer with heat generation from a wall. For this problem, there is no heat flux at $= 0$. Hence $0 \ll x \ll L$ constitutes a half plane and the domain R.

Energy balance on R is given (per unit area) by

Input + Generation = Output + Accumulation

$$0 \quad + \quad g^{+}L \qquad = q|_{x=L} + \frac{d}{dt}\left[\int_0^L \rho C_p (T - T_\infty)dx\right]$$

The balance is

$$\frac{d}{dt}\left[\int_0^L \rho C_p (T - T_\infty)dx\right] - g^{+}L + q|_{x=L} = 0$$

The boundary conditions are

$x = 0 \qquad \frac{dT}{dx} = 0$ (insulated surface)

$x = L \qquad -k\frac{dT}{dx}\Big|_{solid} = h(T_{x=L} - T_\infty)$ (convective)

Where $T_\infty \ll T_{x=L} \ll T_{ss}$; T_{ss} is steady state surface temperature at $t = \infty$.

Define $\theta(x,t) = T(x,t) - T_\infty = X(x)\Gamma(t)$ (separation of variables).

From this, $\Gamma(0) = 0$ and $\Gamma(\infty) = constant$, assume 1.

When $\Gamma(\infty) = 1$, $X(x)$ is the steady state profile.

The steady state solution can be obtained from the following conditions:

$x = 0 \qquad \frac{dT}{dx} = \frac{dX}{dx} = 0$;

$x = L \qquad T = T_{ss}$ or $\theta = T_{ss} - T_\infty = X_{ss}$;

$x = L \qquad -k\frac{dT}{dx}\Big|_{solid} = g^{+}L$ or $\frac{dX}{dx} = -\frac{g^{+}L}{k}$;

Assume a profile $X = a + bx + cx^2$.

Apply the conditions to get

$$X(x) - X_{ss} = \frac{g^{+}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) = T(x,\infty) - T_{ss} \quad \text{or} \quad X(x) = X_{ss} + \frac{g^{+}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right);$$

But at steady state, the convective b.c. also holds.

That is

$$-k\left.\frac{dT}{dx}\right|_{x=L} = -k\left.\frac{dX}{dx}\right|_{x=L} = h(T_{ss} - T_\infty) = hX_{ss} \quad \text{or}$$

$$g^+L = hX_{ss} \Rightarrow X_{ss} = \frac{g^+L}{h};$$

Substitute

$$X(x) = \frac{g^+L}{h} + \frac{g^+L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) = \frac{g^+L^2}{2k}\left(1 - \frac{x^2}{L^2} + \frac{2k}{hL}\right)$$

The transient part of the process is

$$\theta(x,t) = T(x,t) - T_\infty = X(x)\Gamma(t) = \frac{g^+L^2}{2k}\left(1 - \frac{x^2}{L^2} + \frac{2}{Bi}\right)\Gamma(t) \;;\; Bi = \frac{k}{hL}$$

This expression is substituted into the integral energy equation

$$\rho C_p\left(\frac{g^+L^2}{2k}\right)\frac{d}{dt}\left[\Gamma(t)\int_0^L\left(1 - \frac{x^2}{L^2} + \frac{2}{Bi}\right)dx\right] - g^+L - k\left.\frac{d\theta}{dx}\right|_{x=L} = 0$$

$$\rho C_p\left(\frac{g^+L^2}{2k}\right)\frac{d}{dt}\left[\Gamma(t)L\int_0^1\left(1 - \eta^2 + \frac{2}{Bi}\right)d\eta\right] - g^+L + \Gamma(t)(g^+L) = 0$$

$$\rho C_p\left(\frac{g^+L^2}{2k}\right)\frac{d}{dt}\left[\Gamma(t)L\left(1 - \frac{\eta^3}{3} + \frac{2}{Bi}\eta\right)\right] - g^+L(1 - \Gamma(t)) = 0$$

$$\frac{L^2}{2\alpha}\frac{d}{dt}\left[\Gamma(t)\left(\frac{2}{3} + \frac{2}{Bi}\right)\right] - (1 - \Gamma(t)) = 0$$

$$\left(\frac{1}{3} + \frac{1}{Bi}\right)\frac{L^2}{\alpha}\frac{d\Gamma(t)}{dt} = 1 - \Gamma(t)$$

$$\int_0^\Gamma \frac{d\Gamma(t)}{1-\Gamma(t)} = \frac{3\alpha}{L^2}\left(\frac{Bi}{Bi+3}\right)\int_0^t dt$$

$$-ln(1 - \Gamma)\Big|_0^\Gamma = \frac{3\alpha}{L^2}\left(\frac{Bi}{Bi+3}\right)t$$

$$1 - \Gamma = \exp\left[-\frac{3\alpha t}{L^2}\left(\frac{Bi}{Bi+3}\right)\right]$$

Hence

$$\boxed{\theta(x,t) = T(x,t) - T_\infty = X(x)\Gamma(t) = \frac{g^+L^2}{2k}\left(1 - \frac{x^2}{L^2} + \frac{2}{Bi}\right)\left[1 - \exp\left(-\frac{3\alpha t}{L^2}\left(\frac{Bi}{Bi+3}\right)\right)\right]}$$

b) Given

$T(L,t) = 25°\text{C}\; at\; t = 5\,\text{min}\; or\; 300s$

$Bi = \frac{hL}{k} = \frac{(7{,}5)(12)(10^3)}{0.09} = 1;\;\; L = 12\;mm;$

$$\alpha = \frac{k}{\rho C_p} = \frac{0.09}{160(0.94)(10^3)} = 5.984(10^{-7})m^2/s$$

Substitute

$$10 - (-20) = g^+ \frac{(12)^2(10^{-6})}{2(0.09)}(2)\left(1 - exp\left(-\frac{3(5.984)(10^{-7})300}{(12)^2(10^{-6})}\right)\right)$$

$$30 = g^+(0.0016)(1 - 0.392586)$$

$$\boxed{g^+ = \frac{30}{9.7189(10^{-4})} = 3.0869(10^4)\ W/m^3}$$

c) The steady state temperature at $x = 0$

?

$$\boxed{T(0,\infty) = T_\infty + \frac{g^+L^2}{2k}(3) = -15 + \frac{3}{2}\left(\frac{g^+L^2}{2k}\right) = 59.08°\text{C}}$$

and

$$\boxed{T(L,\infty) = T_\infty + \frac{g^+L^2}{k} = 34.39°\text{C}}$$