ENCH 501 Transport Phenomena University of Calgary Name________________________________

Quiz #5 / Time Allowed: 45 minutes Only a "cheat sheet" is allowed. November 17, 2015 AJ

General scope – This problem is on extending a previous derivation from two to three dimensions.

Motivation/Background – People obliged to be outside in cold weather, either for work (construction and soldiers) or recreation (skiers, snowboarders and campers), have available to them a device that releases heat through chemical reaction for warming hands and feet. One type contains a uniform mixture of iron filings, water, cellulose, vermiculite, activated carbon and salt in a pouch wrapped in plastic. To use, the plastic cover is removed to allow air to diffuse into the mixture. The iron reacts with the air in the presence of moisture held in the cellulose to generate heat. Salt catalyzes the reaction. Carbon is the conductor of heat and vermiculite is an insulator that reduces the rate of heat loss. The unit is discarded after a single use. In another type, super-cooled liquid contained in a pouch (that can have different shapes) is triggered to form crystals. Latent heat is released in the process. The unit can be regenerated by immersion in hot water until all the crystals melt and then allowed to cool slowly in air to its super-cooled state before re-use.

(**10 points**) A heating pad consists of a rectangular, thin skin of a highly conductive material filled clear with sodium acetate tri-hydrate liquid at 15°C. At this temperature, the liquid is highly subcooled as its melting point is 58°C. When triggered by an impact, nucleating sites form uniformly throughout the liquid and crystal particles rapidly grow with release of latent heat. If the pad dimensions are $2L \times 2\lambda \times 2\ell$ in the x, y and z directions, the density of the acetate is **ρ** (whether liquid or solid), the thermal conductivity is ***k***, the heat capacity is $\mathbf{C_p}$ and the rate of heat generation per unit volume is $\mathbf{g^+}$, derive an expression for T(x,y,z,t) by the **integral method** for the heating pad. The heat transfer coefficient external to all surfaces of the pad is assumed to be very large and the ambient is at 15°C. Show your steps.
Hints: $\theta(x,y,z,t) = (L^2-x^2)(\lambda^2-y^2)(\ell^2-z^2)\Gamma(t)$ and heat conducted out at x=L (for example) varies with the location of the area dydz.

(Bonus **2 points**) If the pad is 12 cm x 5 cm x 2 cm in dimension, the density of the acetate is 1450 kg/m^3, the heat capacity is 1.379 kJ/kg K, the thermal conductivity is 0.43 W/mK and the rate of heat generation is given as 220 kJ/m^3 s, estimate the temperature at the geometric centre of the pad 3 minutes after nucleation was initiated.

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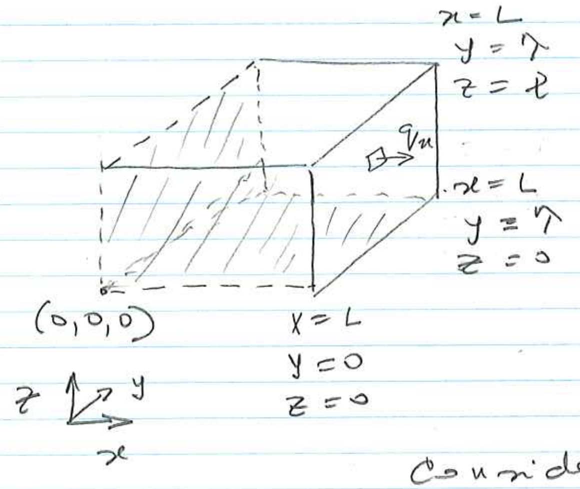

Heat generated within the phase change material (PCM) is both stored within the body and conducted outwards.

Consider a $\frac{1}{8}$th segment of the pad.

The energy balance equation

Input + Generation = Output + Accumulation

$$\text{(1)} \quad g^+ L \ell \lambda = \iint q_{x=L}\, dy\, dz + \iint q_{y=\lambda}\, dx\, dz + \iint q_{z=\ell}\, dx\, dy + \frac{d}{dt}\left[\iiint \rho C_p (T - T_\alpha)\, dx\, dy\, dz\right]$$

(Input $\to 0$)

The temperature profile is assumed to be

$$\text{(2)} \quad \theta = T - T_\alpha = (L^2 - x^2)(\lambda^2 - y^2)(\ell^2 - z^2)\, \Gamma(t)$$

where $T = T_\alpha$ at the surfaces of the pad.

The energy balance equation is

$$g^+ L \ell \lambda = \iint -k \frac{dT}{dx}\bigg|_{x=L} dy\, dz + \iint -k \frac{dT}{dy}\bigg|_{y=\lambda} dx\, dz$$

$$\iint -k \frac{dT}{dz}\Big|_{z=l} dx\,dy + \frac{d}{dt}\left[\iiint \rho c_p (T-T_a)\,dx\,dy\,dz\right]$$

$$\frac{\partial T}{\partial x} = -2x(\lambda^2-y^2)(l^2-z^2)\Gamma(t)$$

$$\frac{\partial T}{\partial y} = -2y(L^2-x^2)(l^2-z^2)\Gamma(t)$$

$$\frac{\partial T}{\partial z} = -2z(L^2-x^2)(\lambda^2-y^2)\Gamma(t)$$

The conditions for the problem are

$$\Gamma(0) = \theta$$

$x=0$ $\quad \frac{\partial \theta}{\partial x} = \frac{\partial T}{\partial x} = 0 \qquad x = L \quad \theta = 0$

$y=0$ $\quad \frac{\partial \theta}{\partial y} = \frac{\partial T}{\partial y} = 0 \qquad y = \lambda \quad \theta = 0$

$z=0$ $\quad \frac{\partial \theta}{\partial z} = \frac{\partial T}{\partial z} = 0 \qquad z = l \quad \theta = 0$

and they are satisfied by thee equations

on substitution for θ and the derivatives

$$\frac{g^+ L l \lambda}{k} = \Gamma \int_{z=0}^{l}\int_{y=0}^{\lambda} 2L(\lambda^2-y^2)(l^2-z^2)\,dy\,dz \quad +$$

$$\Gamma \int_{z=0}^{l} \int_{x=0}^{L} 2\lambda \left(L^2 - x^2\right)\left(l^2 - z^2\right) dx\, dz \quad +$$

$$\Gamma \int_{y=0}^{\lambda} \int_{x=0}^{L} 2l \left(L^2 - x^2\right)\left(\lambda^2 - y^2\right) dx\, dy \quad +$$

$$\frac{\rho C_p}{k} \frac{d}{dt} \left[\Gamma \int_{z=0}^{l} \int_{y=0}^{\lambda} \int_{x=0}^{L} \left(L^2 - x^2\right)\left(\lambda^2 - y^2\right)\left(l^2 - z^2\right) dx\, dy\, dz \right]$$

(3)

$$\frac{g^{+} L l \lambda}{k} = \Gamma \left[\left(2L \cdot \frac{4}{9} \lambda^3 l^3\right) + \left(2\lambda \cdot \frac{4}{9} L^3 l^3\right) + \left(2l \cdot \frac{4}{9} L^3 \lambda^3\right) \right] + \frac{1}{\alpha} \frac{d}{dt} \left\{ \Gamma \left[\frac{8}{27} L^3 \lambda^3 l^3 \right] \right\}$$

$$\frac{27}{8} \frac{g^{+}}{k} \frac{1}{L^2 \lambda^2 l^2} = \Gamma \, \frac{27}{8} \cdot \frac{8}{9} \left(\frac{1}{L^2} + \frac{1}{\lambda^2} + \frac{1}{l^2} \right) + \frac{1}{\alpha} \frac{d\Gamma}{dt}$$

or

(4) $$\frac{1}{\alpha} \frac{d\Gamma}{dt} + \beta \Gamma = \delta$$

where $\beta = 3 \left(\frac{1}{L^2} + \frac{1}{\lambda^2} + \frac{1}{l^2} \right)$ and $\delta = \frac{27}{8} \frac{g^{+}}{k} \frac{1}{L^2 \lambda^2 l^2}$.

Let $Y = \beta\Gamma - \delta$ $\qquad dY = \beta\, d\Gamma$

$$\frac{1}{\alpha\beta}\frac{dY}{dt} = -Y \qquad \text{or} \qquad \frac{dY}{Y} = -\alpha\beta\, dt$$

$$\int_{-\delta}^{\beta\Gamma-\delta}\frac{dY}{Y} = -\alpha\beta\int_0^t dt$$

$$\ln\left(\frac{\beta\Gamma-\delta}{-\delta}\right) = -\alpha\beta t$$

$$-\frac{\beta\Gamma}{\delta} + 1 = \exp(-\alpha\beta t)$$

$$\Gamma = \frac{\delta}{\beta}\left(1 - e^{-\alpha\beta t}\right)$$

$$\therefore\ T(x,y,z,t) - T_a = (l^2 - x^2)(\lambda^2 - y^2)(l^2 - z^2)\cdot\frac{\delta}{\beta}\cdot\left(1 - e^{-\alpha\beta t}\right)$$

Bonus Q.

At the geometric centre of pad,

$x = 0$, $y = 0$ and $z = 0$

Evaluate parameters,

$$\delta = \frac{27}{8}\frac{g^+}{K}\frac{1}{L^2\lambda^2 l^2} = \frac{27}{8}\frac{(220)(10^3)}{0.43\,(0.06\times0.025\times0.01)^2}$$

$$\delta = 7.674419\,(10^{15})$$

$$\beta = 3\left(\frac{1}{L^2} + \frac{1}{\lambda^2} + \frac{1}{l^2}\right) = 3\left[\frac{1}{(0.06)^2} + \frac{1}{(0.025)^2} + \frac{1}{(0.01)^2}\right]$$

$$= 35{,}633.333$$

$$\alpha = \frac{k}{\rho c_p} = \frac{0.43}{1450(1379)} = 2.15048\,(10^{-7})\ m^2/s$$

on substitution

$$T(0,0,0,180_s) - T_a = 48.4587\,(0.7483)$$

$$= 36.259\ ^\circ C$$

$$\therefore\quad T(0,0,0,180_s) \doteq 51.3^\circ C \longrightarrow$$