ENCH 501 Transport Phenomena **Quiz #5** **Name** ______________________

Time Allowed: 45 minutes

A long, rectangular bar of yellow brass, an alloy of copper (65%) and zinc, has a cross-section of 18 cm by 10 cm. It was heated to a uniform temperature of 820°C in the process of annealing. Then it was dipped into a large reservoir of water at a uniform temperature of 7°C to quench (or quickly reduce its temperature). As soon as the metal touches the water, the water in the region near the bar starts to boil at and maintain a temperature of 96°C (in Calgary). The heat transfer coefficient around the bar can be assumed to be very large.

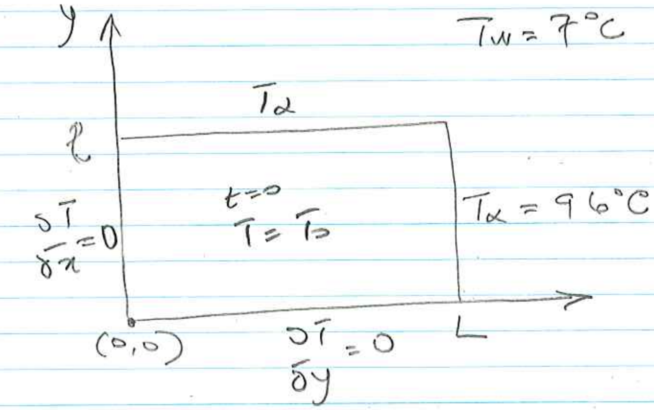
a) Use the **integral method** to derive an expression for the temperature profile T(x, y, t) at a cross-section of the bar. Show all your steps. State any assumptions made.

b) After how long will the temperature at the geometric centre of the bar be at a temperature of 250°C?

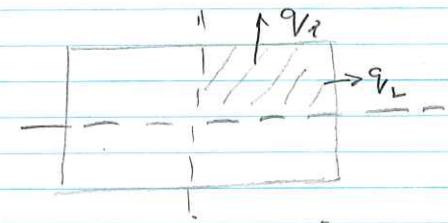
Data: Properties of brass – ρ = 8,800 kg/m^3; k = 119 W/m K; C_p = 0.38 kJ/kg K

Hint: For the accumulation term, find the energy content of a volume ($dx \cdot dy \cdot 1$) and integrate in both x and y directions. Then d/dt. You may assume a profile such as
$T(x, y, t) - T_\alpha = X(x)Y(y)\Gamma(t) + (T_o - T_\alpha)e^{-1000t}$, where $\Gamma(0) = 0$.

ENCH 501 Quiz #5 Solution November 6, 2012



Consider a quadrant of the cross-section



L = 9 cm and l = 5 cm.

There is no generation of heat inside the bar, h is assumed infinite and the boiling water is at 96°C = T_α. This will be the temperature at the surface of the bar, not T_w = 7°C.

Energy balance - per unit length of bar

$$\cancel{\text{Input}}_{0} + \cancel{\text{Gen}}_{0} = \text{Output} + \text{Accum}$$

$$\text{Output} = \int_0^l q\Big|_{x=L} dy + \int_0^L q\Big|_{y=l} dx$$

$$= \int_0^l \left(-k \frac{dT}{dx}\Big|_{x=L}\right) dy + \int_0^L \left(-k\frac{dT}{dy}\Big|_{y=l}\right) dx.$$

Accumulation:

$$\frac{d}{dt}\left[\int_0^l\int_0^L (T-T_\alpha)\, C_p\, \rho\, dx\, dy\right]$$

The integral energy equation is:

$$\int_0^{l} k \frac{dT}{dx}\bigg|_L dy + \int_0^{L} k \frac{dT}{dy}\bigg|_l dx$$

$$= \frac{d}{dt}\left[\int_0^{l}\int_0^{L} \rho C_p (T - T_\alpha)\, dx\, dy\right.$$

Boundary Conditions

$x = 0 \qquad \frac{dT}{dx} = 0 \qquad$ symmetry

$x = L \qquad T = T_\alpha$

$y = 0 \qquad \frac{dT}{dy} = 0 \qquad$ symmetry.

$y = l \qquad T = T_\alpha$

Assume a profile: even function for x and y

$$\theta(x, y, t) = T(x, y, t) - T_\alpha = (L^2 - x^2)(l^2 - y^2)\Gamma(t) + (T_0 - T_\alpha)e^{-1000t}$$

This equation satisfies all the boundary conditions, and the initial condition that at $t = 0$, $T = T_0$ if $\Gamma(0) = 0$

The second term on the r.h.s. rapidly decays to zero for $t > 0$. Hence at $x = L$ and $y = l$, $T \simeq T_\alpha$ for $t > 0$.

$$\left.\frac{dT}{dx}\right|_L = \left. -2x\left(l^2-y^2\right)\Gamma(t)\right|_L = -2L\left(l^2-y^2\right)\Gamma(t)$$

$$\left.\frac{dT}{dy}\right|_l = \left. -2y\left(L^2-x^2\right)\Gamma(t)\right|_l = -2l\left(L^2-x^2\right)\Gamma(t)$$

Substitute into energy equation

$$\int_0^l k(-2L)(l^2-y^2)\Gamma(t)\,dy + \int_0^L k(-2l)(L^2-x^2)\Gamma(t)\,dx$$

$$= \frac{d}{dt}\left[\int_0^l\int_0^L \rho C_p\left\{(L^2-x^2)(l^2-y^2)\Gamma(t) + (T_0-T_\infty)e^{-1000t}\right\}dx\,dy\right]$$

Divide through by ρC_p and let $\alpha = k/\rho C_p$

$$(-2L)\alpha\Gamma(t)\left(\tfrac{2}{3}l^3\right) + (-2l)\alpha\Gamma(t)\left(\tfrac{2}{3}L^3\right)$$

$$= \frac{d}{dt}\left[\Gamma\left(\tfrac{2}{3}L^3\right)\left(\tfrac{2}{3}l^3\right) + (T_0-T_\infty)e^{-1000t}Ll\right]$$

$$-3\alpha\left[\frac{1}{L^2}+\frac{1}{l^2}\right]\Gamma(t) = \frac{d\Gamma}{dt} + \frac{9}{4}\frac{(T_0-T_\infty)}{L^2l^2}\frac{d\left(e^{-1000t}\right)}{dt}$$

or $$\frac{d\Gamma}{dt} = \beta\Gamma(t) + \varepsilon e^{-1000t}$$

where $$\beta = -3\alpha\left[\frac{1}{L^2}+\frac{1}{l^2}\right]$$

and $$\varepsilon = \frac{+9000}{4}\frac{(T_0 - T_a)}{L^2 l^2}$$

Solve equation

$$e^{\int -\beta dt}\frac{d\Gamma}{dt} + e^{\int -\beta dt}(-\beta)\Gamma = e^{\int -\beta dt}\cdot \varepsilon e^{-1000t}$$

$$\Gamma e^{\int -\beta dt} = \varepsilon \int e^{\int -\beta dt} e^{-1000t} dt + C$$

$$\Gamma e^{-\beta t} = \varepsilon \int e^{-(\beta + 1000)t} dt + C$$

$$\Gamma e^{-\beta t} = \frac{\varepsilon}{-(\beta + 1000)} e^{-(\beta + 1000)t} + C$$

when $t = 0$ $\Gamma = 0$ $\therefore$ $C = \dfrac{\varepsilon}{\beta + 1000}$

$$\therefore \Gamma(t) = \frac{\varepsilon}{\beta + 1000}\left(e^{\beta t} - e^{-1000t}\right)$$

Hence

$$T(x,y,t) - T_a = (L^2 - x^2)(l^2 - y^2)\frac{\varepsilon}{\beta + 1000}\left(e^{\beta t} - e^{-1000t}\right) +$$

$$(T_0 - T_a)\, e^{-1000t}$$

(b) At the geometric centre, $x = 0$, $y = 0$

$$T(0,0,t) - T_\alpha = 250 - 96 =$$

$$\frac{-(9000/4)(T_0 - T_\alpha)}{-3\alpha\left[\frac{1}{L^2} + \frac{1}{l^2}\right] + 1000}\left(e^{-3\alpha\left[\frac{1}{L^2}+\frac{1}{l^2}\right]t} - e^{-1000t}\right)$$

$$+ (T_0 - T_\alpha)e^{-1000t}$$

$$\alpha = \frac{k}{\rho c_p} = \frac{119}{8800(380)} = 3.5586(10^{-5})$$

$$-3\alpha\left(\frac{1}{L^2} + \frac{1}{l^2}\right) = -5.5883(10^{-2})$$

$\therefore$

$$250 - 96 \doteq \frac{9000(820-96)}{4[(5.5883)(10^{-2}) + 1000]} e^{-5.5883(10^{-2})t}$$

$$t \simeq 42.2 \text{ s}$$