ENCH 501 F2018 **Quiz #4** Name:______________________________________

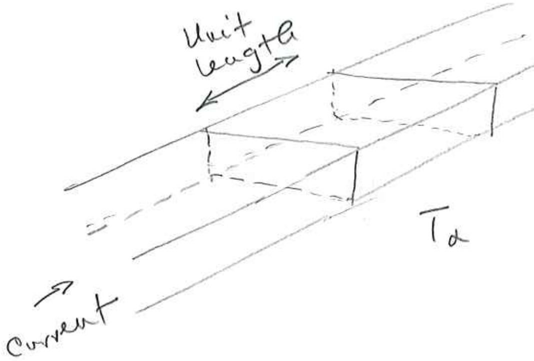
November 6, 2018 Time Allowed: 45 minutes aJ

The cross-section of a long metal bar is rectangular, and the bar is initially at the same temperature as the ambient air at $\mathbf{T_\alpha}$. The flow of an electric current is suddenly started through the bar so that heat is generated uniformly and at a constant rate per unit volume **g+** within the bar. If the bar cross-section is 2L (along the x-direction) by 2ℓ (along the y-direction), obtain a relationship for **T(x,y,t)** at the cross-section by the **integral** method. Assume that the heat transfer coefficient at the surfaces of the bar **h** is infinite and the properties of the metal – **ρ**, **Cp** and **k** are constants. <u>Show important steps</u>.

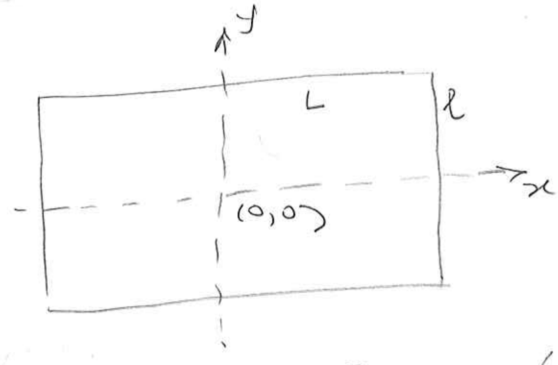
Hints: Consider a unit length of the bar. Until steady state, some of the heat generated will be stored in the bar. At any instant, the energy content of a unit length of the bar may be obtained by first considering the energy content of a **dx** by **dy** by **1** volume of the bar. Then all the energy for similar elements at the cross-section are added. A double integral is in the accumulation term.

Do not spend much time re-deriving what is in your notes. Use any relevant previous results but give enough steps that a reader can follow your analysis.

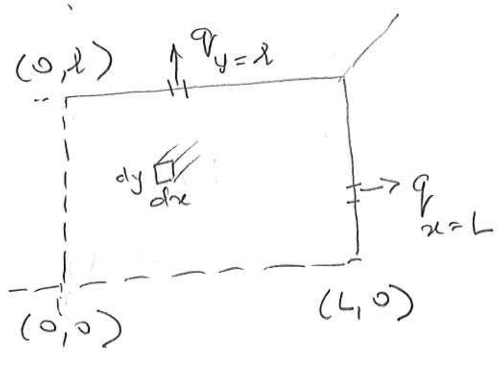
ENCH 501   Quiz 4 Solution   Nov. 6, 2018   GJ



The cross section is like



The temperature field $T(x, y, t)$ will be determined for a quadrant as the lines $x = 0$ and $y = 0$ represents planes of symmetry.



The energy balance equation for the control volume (the quadrant with unit length along the bar) by the integral method is

$$\underbrace{\text{Input}}_{0} + \text{Generation} = \text{Output} + \text{Accumulation}$$

$$\text{Generation rate} = g^{+} L l$$

$$\text{Output rate} = \int_0^l q_{x=L}\, dy + \int_0^L q_{y=l}\, dx$$

$$\text{Accumulation} = \frac{d}{dt}\left[\int_0^l \int_0^L \rho\,(dx\,dy)\, C_p \left(T(x,y,t) - T_a\right)\right]$$

The integral ~~energy~~ equation is

$$g^{+} L l = \int_0^l -k \left.\frac{\partial T}{\partial x}\right|_{x=L} dy + \int_0^L -k \left.\frac{\partial T}{\partial y}\right|_{y=l} dx +$$

$$\frac{\partial}{\partial t}\left[\int_0^l \int_0^L \rho C_p (T - T_\alpha)\, dx\, dy \right] \quad (1)$$

Let $\theta(x, y, t) = T(x, y, t) - T_\alpha$

Also, let the solution be separable, i.e.

$$\theta(x, y, t) = X(x)\, Y(y)\, \Gamma(t)$$

where $X(x)$ and $Y(y)$ have to be even functions to satisfy the symmetry conditions.

At long times, the heat generated equals the heat output and accumulation equals zero. Hence $X(x)\,Y(y)$ is the steady state solution as $\Gamma(\alpha)$ equals a constant, say 1.

At short times, $T(x, y, t) = T_\alpha \Rightarrow \Gamma(0) = 0$

Assume that

$$\theta(x, y, t) = a_0 (L^2 - x^2)(l^2 - y^2)\, \Gamma(t) \quad (2)$$

This satisfies the conditions –

$x = 0 \quad \frac{\partial T}{\partial x} = 0 \quad$ (symmetry)

$y = 0 \quad \frac{\partial T}{\partial y} = 0 \quad$ ✓

$x = L \quad T = T_a \quad (h \to \infty)$

$y = l \quad T = T_a \quad$ ✓

The steady state solution (see Notes) is obtained by substituting the profile

$$\theta(x, y, \infty) = a_0 (L^2 - x^2)(l^2 - y^2)$$

into the integral energy equation without the accumulation term. This gives

$$a_0 = \frac{3}{4} \frac{g^+/k}{L^2 + l^2} \qquad (3)$$

The gradients in the output terms are obtained as

$$\left.\frac{\partial T}{\partial x}\right|_{x=L} = -2 L a_0 (l^2 - y^2)\, \Gamma(t) \qquad (4\,a,b)$$

$$\left.\frac{\partial T}{\partial y}\right|_{y=l} = -2 l a_0 (L^2 - x^2)\, \Gamma(t)$$

Substitute equations 2 and 4 into equation 1, and simplify

$$g^{+} L l = 2 k L a_0 \Gamma(t) \int_0^{l} (l^2 - y^2)\, dy \;+$$

$$2 k l a_0 \Gamma(t) \int_0^{L} (L^2 - x^2)\, dx \;+$$

$$\frac{\partial}{\partial t} \left[\int_0^{l} \int_0^{L} \rho C_p a_0 (L^2 - x^2)(l^2 - y^2) \Gamma(t)\, dx\, dy \right] \qquad (5)$$

$$g^{+} L l = \frac{4}{3} k a_0 (L l^3 + l L^3) \Gamma(t) \;+$$

$$\rho C_p a_0 \frac{d}{dt} \left[\Gamma(t) \cdot \frac{4}{9} L^3 l^3 \right]$$

This equation is of the form

$$\frac{d\, \Gamma(t)}{dt} + a \Gamma(t) = b \qquad (6)$$

where $a = \dfrac{4/3\, k a_0 (L l^3 + l L^3)}{4/9\, L^3 l^3 \rho C_p a_0}$

and $b = \dfrac{g^{+} L l}{\frac{4}{9} L^3 l^3 \rho C_p a_0}$

both constants

This equation is subject to the condition $t = 0 \quad \Gamma = 0$

Re-arrange equation

$$d \ln (b - a\Gamma(t)) = -a\,dt$$

or $$\ln (b - a\Gamma) = -at + c$$

Apply condition

$$c = \ln b$$

$$\therefore \ln \left(\frac{b - a\Gamma}{b} \right) = -at$$

or $$1 - \frac{a}{b}\Gamma = e^{-at}$$

$$\therefore \Gamma(t) = \frac{b}{a}\left(1 - e^{-at}\right) \qquad (7)$$

Substitute equations (3), and (7) into equation (2) to obtain the transient profile

$$\theta(x,y,t) = T(x,y,t) - T_a = a_0 (l^2 - x^2)(l^2 - y^2) \cdot \frac{b}{a}\left(1 - e^{-at}\right)$$

$\longrightarrow$