GJ
2014

University of Calgary
Department of Chemical & Petroleum Engineering

ENCH 501: Transport Processes

Final Examination, Fall 2006

Time: 3.30 - 6.30 pm **Thursday, December 14, ~~2005~~ 6**

Instructions: **Attempt All Questions.**
Use of Electronic Calculators allowed.
Open Notes, Open Book Examination.

Problem #1 (30 points)

Canada's National (CN) tower in Toronto is the world's tallest building and free standing structure, at 553.33m. At the level of 342m is the glass floor on which visitors can stand and look straight down. The tempered and hardened glass which covers an area of 23.8m^2 is a layered structure and it is 6.35cm (or 2.5 inches) thick. (For the purposes of this problem, the glass will be assumed to be one solid piece or slab.) The tower often experiences high winds and thus the convective heat transfer coefficient on the outside surface of the glass may be assumed to be large. We shall also assume that, overnight, the heat inside the observation deck at the glass floor level is turned off so that the glass attains the temperature of the air outside. On cold days, when the workers and the first visitors arrive, vapour from their breath (like in the car) condenses on the glass and obscures the view. To clear the view, a fan above the glass floor blows air at 25°C (T_1) over the surface. The heat transfer coefficient at the inside surface is also assumed to be large.

On a clear winter but windy morning, the temperature outside the tower (T_∞) was measured as -15°C. The glass floor was at this temperature uniformly across its thickness. At t = 0, the fan was turned on to blow warm air over the surface inside the observation deck.

a) Use the ***integral method*** to derive relationships for temperature profiles across the glass wall which is assumed to be an infinite wall, i.e. obtain T(x,t). Show all your steps.
b) Estimate how long it will take for the temperature at the mid-plane of the glass to reach 4.5°C.

Hints and Data:
Properties of tempered glass: ρ = 2270 kg/m^3; **C_p** = 0.745 kJ/kg K; **k** = 1.38 W/mK

You may find it convenient to do the problem in two parts - stage 1 when the penetration depth is less than the wall thickness (L) and stage 2 from this point to steady state. You may find the following relationship useful for the second stage:

$$\frac{T - T_\infty}{T_1 - T_\infty} = (1 - \frac{x}{L}) - \Gamma(t)\frac{x}{L}(1 - \frac{x}{L})$$; where x is distance from the inside surface.

Check first if this expression satisfies the temperature profile at the start and end of stage 2.

Problem #2 (30 points)

A thin slab (10cm x 6cm x 0.2cm) of camphor or naphthalene (used as moth and cockroach repellent) is suspended in an air steam with the long side parallel to the direction of air flow. The naphthalene sublimates (changes phase directly from solid to vapour) and its vapour pressure at the room condition is 11Pa. The ambient pressure is 89 kPa and the air temperature is 25°C. The edges of the slab are sharp and both momentum and concentration boundary layers develop over the two large surfaces.

a) Derive an expression for the steady concentration profile for the naphthalene in the boundary layer of the gas phase. Use ***the integral*** method and show all your steps.

b) If the air flows at 1.2m/s, estimate the rate at which the naphthalene is being removed from the slab. Assume the edges are covered.

For the problem, you may neglect the convective component of mass or molar flux. The concentration of the naphthalene at the surface of the solid may be derived from Raoult's law and there is negligible naphthalene in the ambient air.

Data: Properties of naphthalene: ρ_{solid} = 1,100 kg/m^3; Molar mass = 128.2 kg/kmol; D_{AB} (naphthalene in air) = 6.1 (10^{-6}) m^2/s;
Properties of air: ρ = 1.178 kg/m^3; μ = 1.85(10^{-2}) mPa s

The Universal Gas constant, $\mathcal{R}$= 8.316 J/mol K

Problem #3 (20 points)

In angioplasty, a catheter is treaded from the thigh into a large blood vessel and guided up to the heart when arteries serving the heart are partially or nearly completely occluded. Since blood continues to flow during the procedure, the system approximately models couette flow between concentric cylinders. (A similar system is noted when a "pig" is sent through a petroleum pipeline to scrape off deposits at the wall.)

Consider a section of the flow system where the catheter, radius κR, is moving at velocity U_o in a direction opposite to that for the blood flow in the annular space, with the radius of the inner surface of the outside cylinder equal R. The pressure gradient along the flow direction is constant and the blood vessel/catheter are inclined at angle θ to the horizon.

a) Assume blood is Newtonian (density ρ and viscosity μ), derive a function for the velocity profile across the annular space. Show your steps.

b) Under what conditions will the net flow of blood across any plane through the annular space equal zero?

Problem #4 (20 points)

The Athabasca river cuts through one of the large oil sands deposits in Alberta and, in some sections, droplets of oil are dislodged and float down the river. Sand grains are also carried and deposited. Like other alluvial rivers, the river forms a stable channel and transport of particles and sediments occur without a net deposition or degradation. That is, as long as the flow rate of water is steady, the local depths (H) and width (W) of the channel remains unchanged over decades.

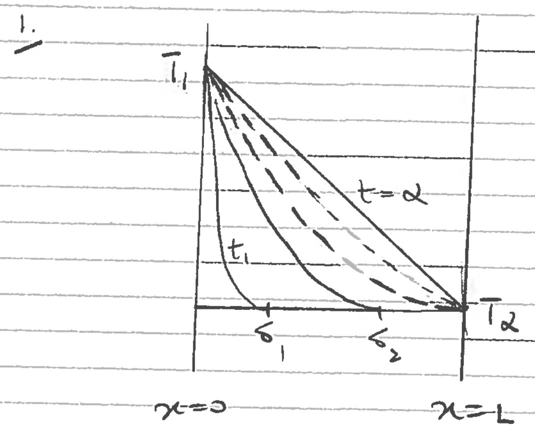
It is given that the width of the river (W) is related to other parameters, i.e.

W = f(Q, ρ, γ_s, μ, g, D, S, H) where Q is volumetric flow rate, ρ is the density of the water, $\gamma_s = (\rho_s - \rho)g$ where ρ_s is the density of sediment, μ is dynamic viscosity of water at local temperature, g is acceleration of gravity, D is diameter of particles and droplets, S is the local slope (m/m) of the terrain and H is the maximum depth of the river.

What dimensionless groups describe the width-to-depth ratio (W/H) for the flow channel?

ENCH 501 FINAL EXAM Solution Dec. 14, 2006

1.



Since h is large on both sides of the glass, the surface temperatures are the same as for the air around them.

The problem is divided into 2 parts –

Stage 1 – semi-infinite domain $\delta < L$ ~~and there~~ is no heat output from domain.

In Stage 2 – some heat leaves the domain at $x=L$ and temperature profiles adjust until it reaches the steady state profile at $t=\alpha$.

(a)

Stage 1 Initial temp. in glass $= T_\alpha$ and surface at $x=0$ is suddenly raised to and maintained at T_1.

Energy balance: Input + $\cancel{\text{Gen}}^{\,0}$ = $\cancel{\text{Output}}^{\,0}$ + Accum.

$$-k\left.\frac{dT}{dx}\right|_{x=0} = \frac{d}{dt}\left[\int_0^L \rho C_p (T-T_\alpha)\,dx\right]$$

Integral Energy Eq.

$$= \frac{d}{dt}\left[\int_0^\delta \rho C_p T\,dx\right] - \rho C_p T_\alpha \frac{d\delta}{dt}$$

The boundary conditions are

$x=0$ $T=T_1$
$x=\delta(t)$ $T=T_\alpha$
$x=\delta(t)$ $\frac{dT}{dx}=0$

Assume a profile, e.g. $T = a + bx + cx^2$
and apply the b.c.s, one obtains

$$\frac{T - T_\alpha}{T_1 - T_\alpha} = \left(1 - \frac{x}{\delta}\right)^2$$

Substitute this into the integral energy equation, and solve (as on p. 105 Notes)

$$\delta = \sqrt{12 \alpha t}$$

Stage 2 When $\delta = L$, heat starts to be discharged into the cold air outside.

The initial temperature profile is given above, or

$$\frac{T - T_\alpha}{T_1 - T_\alpha} = \left(1 - \frac{x}{L}\right)^2$$

The final temperature profile is the steady state or

$$\frac{T - T_\alpha}{T_1 - T_\alpha} = \left(1 - \frac{x}{L}\right)$$

The suggested profile

$$\frac{T - T_\alpha}{T_1 - T_\alpha} = \left(1 - \frac{x}{L}\right) - \Gamma(t) \frac{x}{L}\left(1 - \frac{x}{L}\right)$$

satisfies the above profiles if

$$\Gamma(0) = 1 \qquad \text{and} \qquad \Gamma(\infty) = 0$$

What is required is to obtain $\Gamma(t)$.

The energy balance: Input + Gen = Output + Accum
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\downarrow 0$

$$-k\left.\frac{dT}{dx}\right|_{x=0} = -k\left.\frac{dT}{dx}\right|_{x=L} + \frac{d}{dt}\left[\int_0^L \rho C_p (T - T_a)\, dx\right]$$

From the profile

$$\frac{dT}{dx} = \left(-\frac{1}{L} - \frac{\Gamma}{L} + \frac{2\Gamma x}{L^2}\right)(T_1 - T_a)$$

at $x = 0$ $\quad \dfrac{dT}{dx} = -\left(\dfrac{\Gamma + 1}{L}\right)(T_1 - T_a)$

at $x = L$ $\quad \dfrac{dT}{dx} = -\left(\dfrac{1 - \Gamma}{L}\right)(T_1 - T_a)$

Substitute

$$+\left(\frac{1+\Gamma}{L}\right) = \left(\frac{1-\Gamma}{L}\right) + \frac{1}{\alpha}\frac{d}{dt}\left[\int_0^L \left[\left(1 - \frac{x}{L}\right) - \Gamma\frac{x}{L}\left(1 - \frac{x}{L}\right)\right] dx\right]$$

$$\frac{2\Gamma}{L} = \frac{L}{\alpha}\frac{d}{dt}\left[\int_0^1 \left[(1-\eta) - \Gamma\eta(1-\eta)\right] d\eta\right]$$

$$\frac{2\Gamma\alpha}{L^2} = \frac{d}{dt}\left[\eta - \frac{1}{2}\eta^2 - \frac{\Gamma}{2}\eta^2 + \frac{\Gamma}{3}\eta^3\right]_0^1 = \frac{d}{dt}\left[-\frac{\Gamma}{6}\right]$$

i.e. $\quad \dfrac{d\Gamma}{dt} = -\dfrac{12\alpha}{L^2}\Gamma \quad$ or $\quad \dfrac{d\Gamma}{\Gamma} = -\dfrac{12\alpha}{L^2}dt$

$$\ln \Gamma = -\frac{12\alpha}{L^2}t + C$$

But when $t = 0$, $\Gamma = 1$ $\Rightarrow$ $C = 0$

$$\therefore \quad P = \exp\left[-\frac{12\alpha t}{L^2}\right]$$ where t is time from start of stage 2.

$$\therefore \quad \frac{T - T_\alpha}{T_1 - T_\alpha} = \left(1 - \frac{x}{L}\right) - e^{-\frac{12\alpha t}{L^2}} \frac{x}{L}\left(1 - \frac{x}{L}\right)$$ for stage 2.

→

(b) Total time for mid-plane to reach 4.5°C
= stage 1 time + stage 2 time.

Stage 1 $\quad \delta = L = \sqrt{12\alpha t}$

$$L = 6.35(10^{-2}) \text{ m}$$

$$\alpha = \frac{k}{\rho c_p} = \frac{1.38}{2270(745)} = 8.16(10^{-7}) \frac{m^2}{s}$$

$$\therefore \quad t = 411.78 \text{ s}$$

Stage 2 $\quad x/L = 0.5$

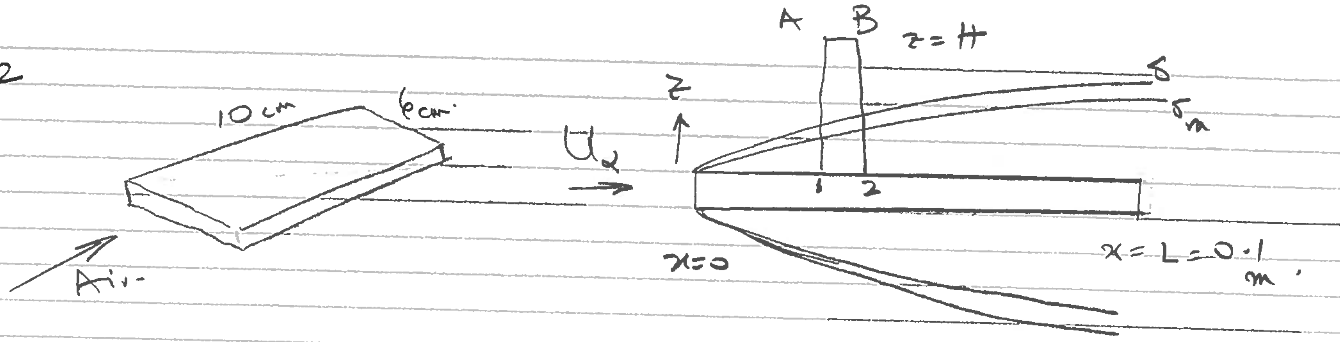
$$\frac{4.5 + 15}{25 + 15} = (1 - 0.5) - e^{-\frac{12\alpha t}{L^2}}(0.5)(1 - 0.5)$$

$$0.4875 = 0.5 - 0.25\, e^{-\frac{12\alpha t}{L^2}}$$

$$\frac{\alpha t}{L^2} = 0.2496 \quad , \quad t = 1,233.61 \text{ s}$$

$\therefore$ total time $= 411.78 + 1,233.61 = 1,645.4$ s
(27.42 min) →

#2



Consider a differential element 12AB

The momentum balance is as derived in the Notes, p 85ff.

The momentum integral eq. is 5.13.

$$\mu \frac{du}{dz}\Big|_{z=0} = \frac{d}{dx}\left[\int_0^{\delta} \rho (U_\infty - u) u\, dy\right] \qquad (1)$$

Use conditions

$$\left.\begin{array}{ll} z = 0 & u = 0 \\ z = \delta & u = U_\infty \\ z = \delta & \partial u/\partial z = 0 \\ z = 0 & \partial^2 u/\partial y^2 = 0 \end{array}\right\} \text{ as in Notes} \qquad (2)$$

and assume $u = a + bz + cz^2 + dz^3$ (3)

we obtain

$$\frac{u}{U_\infty} = \frac{3}{2}\left(\frac{z}{\delta}\right) - \frac{1}{2}\left(\frac{z}{\delta}\right)^3 \qquad (4)$$

where $\delta = 4.64\left(\frac{\nu x}{U_\infty}\right)^{1/2}$ from eq. (1)

For the concentration of solute A, the material balance is as in the Notes, p. 123ff. That is

$$N_A\Big|_{z=0} = k_c C\left(y_A\Big|_{z=0} - \underbrace{\cancel{y_{A\infty}}}_{0}\right) = -C D_{AB} \frac{\partial y_A}{\partial z}\Big|_{z=0}$$

$$\frac{d}{dx}\left[\int_0^H C u y_A\, dz\right] - \underbrace{C y_{A\infty} \cancel{\frac{d}{dx}}\left[\int_0^H u\, dz\right]}_{0} \qquad (5)$$

The conditions are

$$z = 0 \qquad y_A = y_{AS} = \frac{P_A^{vP}}{P_t} \quad \text{(Raoult's Law)}$$

$$z = \delta_m \qquad y_A = y_{A\infty} = 0$$

$$z = \delta_m \qquad \frac{dy_A}{\partial z} = 0$$

If we assume $\quad y_A = a + bz + cz^2$

$$\frac{y_A}{y_{AS}} = \left(1 - \frac{z}{\delta_m}\right)^2 \qquad (6)$$

Substitute (4) and (6) into (5)

$$\frac{2 y_{AS} D_{AB}}{\delta_m} = \frac{d}{dx}\left[\int_0^{\delta_m} \left(\frac{3}{2}\frac{z}{\delta} - \frac{1}{2}\left[\frac{z}{\delta}\right]^3\right)\left(1 - \frac{z}{\delta_m}\right)^2 (U_\alpha y_{AS})\, dz\right] \qquad (7)$$

Define $\quad \xi = \delta_m/\delta < 1 \qquad Sc = \frac{\nu}{D_{AB}} = 2.575 > 1$ (from data)

$$\frac{2 \cancel{y_{AS}} D_{AB}}{\delta_m U_\alpha \cancel{y_{AS}}} = \frac{d}{dx}\left[\delta \int_0^{\xi} \left(\frac{3}{2}\eta - \frac{1}{2}\eta^3\right)\left(1 - \frac{\eta}{\xi}\right)^2 d\eta\right] \; ; \; \eta = \frac{z}{\delta}$$

$$\frac{2 D_{AB}}{\xi \delta U_\alpha} = \frac{d}{dx}\left[\delta\left(\frac{1}{8}\xi^2 - \frac{1}{120}\xi^4\right)\right] \qquad (8)$$

$$\frac{\cancel{8 \times} 2 D_{AB}}{U_\alpha} = \xi \delta \frac{d}{dx}\left(\delta \xi^2\right) \quad \text{if } 2^{nd} \text{ term r.h.s is neglected}$$

$$\frac{16 D_{AB}}{U_\alpha} = \xi \delta \left[\delta \frac{d\xi^2}{dx} + \xi^2 \frac{d\delta}{dx} \right]$$

$$\frac{16 D_{AB}}{U_\alpha} = \left[\xi \delta^2 \frac{d\xi^2}{dx} + \xi^3 \delta \frac{d\delta}{dx} \right] \quad (9)$$

But from Notes, eq. 5.18 and 5.19

$$\delta \frac{d\delta}{dx} = \frac{140}{13} \frac{\nu}{U_\alpha} \quad \text{and} \quad \delta^2 = \frac{280}{13} \frac{\nu x}{U_\alpha}$$

$$\frac{16 D_{AB}}{U_\alpha} = \frac{280}{13} \frac{\nu x}{U_\alpha} \xi \frac{d\xi^2}{dx} + \xi^3 \frac{140}{13} \frac{\nu}{U_\alpha}$$

$$4 \frac{D_{AB}}{\nu} = \frac{70}{13} x \xi \frac{d\xi^2}{dx} + \frac{35}{13} \xi^3$$

$$\frac{52}{35} \frac{D_{AB}}{\nu} = 4 x \xi^2 \frac{d\xi}{dx} + \xi^3 \quad \text{where } \xi = \xi(x)$$

$$\frac{52}{35} \frac{D_{AB}}{\nu} = \frac{4}{3} x \frac{d\xi^3}{dx} + \xi^3 \quad (10)$$

Solution is

$$\xi^3 = C x^{-3/4} + \frac{52}{35} \frac{D_{AB}}{\nu} \quad (11)$$

Assume a length x_o at the leading edge without solute

i.e. at $x = x_o$, $\xi = \frac{\delta_m}{\delta} = 0$

$$\therefore \; C \text{ in eq. } (11) = -\frac{52}{35} \frac{D_{AB}}{\nu} x_o^{3/4}$$

$$\therefore \quad \zeta = \left[\frac{52}{35}\frac{D_{AB}}{\nu}\right]^{1/3}\left[1 - \left(\frac{x_0}{x}\right)^{3/4}\right]^{1/3}$$

If $x_0 = 0$, as for this problem,

$$\zeta = \left[\frac{52}{35}\frac{D_{AB}}{\nu}\right]^{1/3} = 1.1411\, Sc^{-1/3}, \text{ independent of } x.$$

The profiles are

$$\therefore \quad \frac{y_A(x,z)}{y_{As}} = \left(1 - \frac{z}{\delta_m}\right)^2$$

where $\delta_m = 1.1411\, Sc^{-1/3}\,(4.64)\left(\frac{\nu x}{u_\infty}\right)^{1/2}$

$\longrightarrow$

(b) The local flux

$$N_A\big|_{z=0} = -c D_{AB}\frac{\partial y_A}{\partial z}\bigg|_{z=0} = +c D_{AB}\left(\frac{2 y_{As}}{\delta_m}\right)$$

Rate of sublimation from slab, two sides

$$\Phi = 2W\int_0^L N_A\big|_{z=0}\, dx \quad ; \quad \begin{array}{l} W = 0.06\ m \\ L = 0.1\ m \end{array}$$

$$= \frac{4WcD_{AB}y_{As}}{\beta}\int_0^L \frac{dx}{x^{1/2}} \quad ; \quad \beta = 5.2945\, Sc^{-1/3}\left(\frac{\nu}{u_\infty}\right)^{1/2}$$

From ~~Data~~

$$Sc = \frac{\nu}{D_{AB}} = 2.5745, \quad \nu = \frac{\mu}{\rho} = 1.5705\,(10^{-5})\ m^2/s$$

$$u_\infty = 1.2\ m/s, \quad y_{As} = \frac{11}{89(10^3)} = 1.236(10^{-4})$$

$$C = \frac{P}{RT} = \frac{89(10^3)}{8.316(273.15+25)} = 35.896 \ \frac{\text{mols}}{m^3}$$

$$Q = \frac{8 W C D_{AB} y_{AS} L^{1/2}}{5.2945 (Sc^{-1/3})(\nu / u_\alpha)^{1/2}} \quad \text{mols}/s$$

$$= \frac{8(0.06)(35.896)(6.1)(10^{-6})(1.236)(10^{-4})(0.1)^{1/2}}{5.2945 (2.5745^{-1/3}) \left(\frac{1.5705(10^{-5})}{1.2}\right)^{1/2}}$$

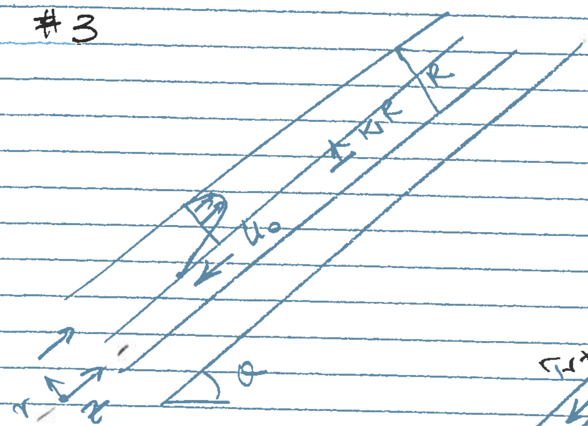
$$= 2.9395 (10^{-7}) \quad \text{mols}/s$$

The mass rate = Q x molar mass

$$= 3.7684 (10^{-5}) \quad g/s$$

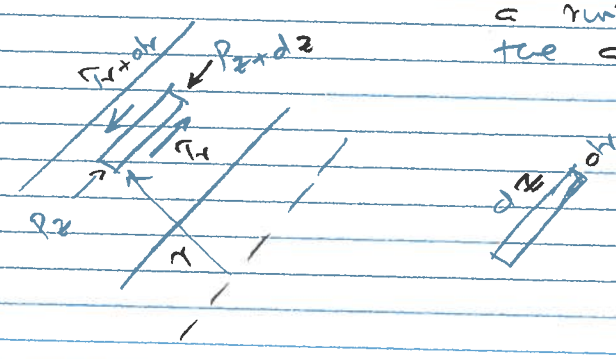
$\longrightarrow$

#3



The catheter is assumed to be moving down while pressure is pushing the fluid upwards.

Consider a differential element - a ring within the annular space.



The force balance is:

$$\left(\tau_r \cdot 2\pi r dz\right)\Big|_r - \left(\tau_r \cdot 2\pi r dz\right)\Big|_{r+dr} + P_z\left(2\pi r dr\right) -$$

$$P_{z+dz}\left(2\pi r dr\right) - \left(2\pi r dr dz\right)\rho g \sin\theta = 0$$

or

$$-\frac{d}{dr}\left(\tau_r \cdot 2\pi r dz\right)dr - 2\pi r dr \frac{dP}{dz} dz - 2\pi r dr dz \rho g \sin\theta = 0$$

or

$$\frac{d\left(\tau_r r\right)}{dr} + r\frac{dP}{dz} + r\rho g \sin\theta = 0$$

$$\frac{d\left(\tau_r r\right)}{dr} = -\left[\frac{dP}{dz} + \rho g \sin\theta\right] r$$

Integrate once

$$\tau_r \cdot r = -\left[\frac{dP}{dz} + \rho g \sin\theta\right]\frac{r^2}{2} + C_1$$

Let $\beta = -\left[\frac{dP}{dz} + \rho g \sin\theta\right]$ and fluid is Newtonian

$$-\mu \frac{du}{dr} = \beta \frac{r}{2} + C_1/r$$

Integrate 2^{nd} time

$$u = -\frac{\beta}{\mu}\frac{r^2}{4} - \frac{C_1}{\mu}\ln r + C_2$$

Apply boundary conditions

$r = K_1 R \qquad u = -U_o$

$r = R \qquad u = 0$

$$-U_o = -\frac{\beta}{\mu}\frac{(K_1R)^2}{4} - \frac{C_1}{\mu}\ln K_1R + C_2$$

$$0 = -\frac{\beta}{\mu}\frac{R^2}{4} - \frac{C_1}{\mu}\ln R + C_2$$

Subtract

$$-U_o = -\frac{\beta}{\mu}\frac{R^2}{4}(K_1^2-1) - \frac{C_1}{\mu}\ln K_1$$

$$\therefore C_1 = \frac{\mu}{\ln K_1}\left(U_o - \frac{\beta}{\mu}\frac{R^2}{4}(K_1^2-1)\right)$$

$$\text{and } C_2 = \frac{\beta}{\mu}\frac{R^2}{4} + \frac{1}{\ln K_1}\left(U_o - \frac{\beta}{\mu}\frac{R^2}{4}(K_1^2-1)\right)\ln R$$

$\therefore$

$$u = -\frac{\beta}{\mu}\frac{r^2}{4} - \frac{C_1}{\mu}\ln r + \frac{\beta}{\mu}\frac{R^2}{4} + \frac{C_1}{\mu}\ln R$$

$$= \frac{\beta}{\mu}\frac{R^2}{4}\left(1 - \frac{r^2}{R^2}\right) + \frac{C_1}{\mu}\ln\frac{r}{R}$$

$$u = -\frac{\left[\frac{dP}{dz} + \rho g \sin\theta\right]}{\mu}\cdot\frac{R^2}{4}\left(1 - \frac{r^2}{R^2}\right) + \frac{1}{\ln K_1}\left(U_o - \frac{\beta}{\mu}\frac{R^2}{4}(K_1^2-1)\right)\ln\frac{r}{R}$$

which can be further re-arranged. $\longrightarrow$

(b) The net flow is given by

$$Q = \int_{KR}^{R} 2\pi r u \, dr$$

For zero net flow, $Q = 0$

$$2\pi \int_{KR}^{R} \left[\gamma\left(1 - r^2/R^2\right) + \frac{C_1}{\mu} \ln\left(r/R\right) \right] r \, dr = 0 \quad ; \quad \gamma = \frac{P}{\mu}\frac{R^2}{4}$$

$$\int_{KR}^{R} \gamma r - \gamma \frac{r^3}{R^2} + \frac{C_1}{\mu} r \ln\left(\frac{r}{R}\right) dr = 0$$

$$R^2 \int_{K}^{1} \left[\gamma\left(\frac{r}{R}\right) - \gamma \frac{r^3}{R^3} + \frac{C_1}{\mu} \frac{r}{R} \ln\left(\frac{r}{R}\right) \right] \frac{dr}{R} = 0$$

Let $\eta = r/R$

$$\int_{K}^{1} \left[\gamma \eta - \gamma \eta^3 + \frac{C_1}{\mu} \eta \ln \eta \right] d\eta = 0$$

$$\left[\gamma \frac{\eta^2}{2} - \frac{\gamma \eta^4}{4} + \frac{C_1}{\mu}\left(\frac{\eta^2}{2}\left(\ln \eta - \frac{1}{2}\right)\right) \right]_{K}^{1} = 0$$

$$\frac{\gamma}{2} - \frac{\gamma}{4} + \frac{C_1}{\mu}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) - \frac{\gamma K^2}{2} + \frac{\gamma K^4}{4} - \frac{C_1}{\mu}\left(\frac{K^2}{2}\left(\ln K - \frac{1}{2}\right)\right) = 0$$

$$\frac{\gamma}{4}\left(1 - 2K^2 + K^4\right) - \frac{C_1}{4\mu}\left\{ 1 + 2K^2\left(\ln K - \frac{1}{2}\right) \right\} = 0$$

$$C_1 = \frac{\mu\gamma(1-\kappa^2)^2}{\left(1+2\kappa^2\left(\ln\kappa - \frac{1}{2}\right)\right)} = \frac{\mu}{\ln\kappa}\left(U_0 - \gamma(\kappa^2-1)\right)$$

from earlier.

$$U_0 = \gamma\left[(\kappa^2-1) + \frac{\ln\kappa(1-\kappa^2)^2}{1+2\kappa^2\left(\ln\kappa-\frac{1}{2}\right)}\right]$$

$$U_0 = +\left[\frac{dP}{dz} + \rho g \sin\theta\right]\frac{R^2}{4\mu}(1-\kappa^2)\left[1 - \frac{\ln\kappa(1-\kappa^2)}{1+2\kappa^2\left(\ln\kappa-\frac{1}{2}\right)}\right]$$

This is the relationship between U_0 and the pressure gradient for no net flow

#4

Given $W = f(Q, \rho, \gamma_s, \mu, g, D, S, H)$ ①

By inspection, the following are dimensionless

$$\frac{W}{H}, \quad S, \quad \frac{H}{D} \quad \text{and} \quad \frac{\gamma_s}{\rho g}$$

Since there are 9 variables and 3 dimensions (M, L, t), 6 dimensionless gps can be formed. We already have 4, ∴ 2 remain

If we remove S, γ_s, D and H from the variables, we have

$$W = \phi(Q, \rho, \mu, g)$$ ②

$$\underset{m}{W} \quad \underset{m^3/s}{Q} \quad \underset{kg/m^3}{\rho} \quad \underset{Pa.s \rightarrow kg/m.s}{\mu} \quad \underset{m/s^2}{g}$$

By inspection, one can form groups

$$\frac{Q^2}{gW^5} \quad \text{and} \quad \frac{Q\rho}{W\mu}$$

$$\therefore \frac{W}{H} = F\left(\frac{Q^2}{gW^5}, \frac{Q\rho}{W\mu}, S, \frac{H}{D}, \frac{\gamma_s}{\rho g}\right)$$ ③

$\longrightarrow$

From equation ②, we can use the Pi Theorem.

$$\Pi_1 = Q^a W^b \rho^c g$$

$$\Pi_2 = Q^a W^b \rho^c \mu$$

Using W, Q, ρ as repeating units

For $\Pi_1 = \left(\frac{L^3}{t}\right)^a (L)^b \left(\frac{M}{L^3}\right)^c \frac{L}{t^2}$

$$L \quad 0 = 3a + b - 3c + 1$$
$$M \quad 0 = c$$
$$t \quad 0 = -a - 2$$

$a = -2$
$b = 5$
$c = 0$

$$\therefore \quad \Pi_1 = \frac{gW^5}{Q^2}$$

$$\Pi_2 = \left(\frac{L^3}{t}\right)^a (L)^b \left(\frac{M}{L^3}\right)^c \frac{M}{Lt}$$

$$L \quad 0 = 3a + b - 3c - 1$$
$$M \quad 0 = c + 1$$
$$t \quad 0 = -a - 1$$

$a = -1$
$b = 1$
$c = -1$

$$\Pi_2 = \frac{W\mu}{Q\rho}$$

Thus one obtains the same dimensionless gps

$\longrightarrow$