University of Calgary
Department of Chemical & Petroleum Engineering

ENCH 501: Mathematical Methods in Chemical Engineering

Final Examination, Fall 2004

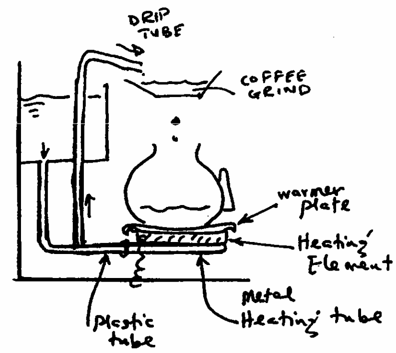
Time: Noon - 3 pm **Thursday, December 16, 2004**

Instructions: **Attempt All Questions.**
Use of Electronic Calculators allowed.
Open Notes, Open Book Examination.

Problem #1 (20 points)

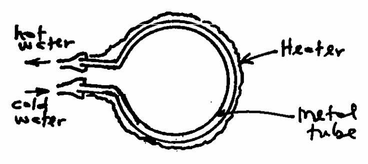
The coffee maker is a simple, yet ingenious device. A fixed volume of water is poured or admitted into a "bucket" or chamber in the device. Water flows, by gravity, through a hole in the bottom of the bucket into a plastic tube attached to a spout. This tube extends to below the "warmer plate" and is joined to an aluminium "heating" tube. The metal tube is coiled under the warmer plate. An electric heating element is bonded between the warmer plate above and the coiled tube below. Thus, as the water flows through, it is heated by the element. (It may be assumed that the heat flux at the wall is constant along the length of the coiled metal tube.) At the outlet, the coiled metal tube is connected to another plastic tube which then conveys the water back up to the drip tap and over coffee grinds placed on a filter in a funnel. The warmer plate on which the brewed-coffee pot is placed is directly under the funnel. For this problem, the focus will only be on the heating of the water by the element.

Water is admitted into the bucket at 15°C. The heating tube has an outside diameter of ½ inch. The wall thickness is 1/16 inches. The total power delivered by the heating element is 500 W and the heat flux along the metal tube wall is assumed uniform, i.e. q_W is constant. The length L of the heating tube is 40 cm and both the velocity and temperature profiles are assumed fully developed.



(a) If the water is heated to a mix-cup or bulk mean value of 45°C at the outlet of the heating tube, estimate the minimum time required to brew a litre of coffee. Identify and justify the assumptions in your analysis.

(b) Estimate the temperature of the wall of the heating tube at a L/2.



Data:
Properties of water (av.) - $\rho = 995.6$ kg.m^3 ; $C_p = 4.186$ kJ/kg K ;
$\mu = 0.8$ mPa s ; k = 0.615 W/m K
1 inch = 25.4 mm Assume k for aluminium is very high.

Problem #2 (30 points)

People are prepared to go to great lengths for a bright-white smile. Tooth enamel whitening is a popular cosmetic procedure. Enamel is the crown or outermost part of a tooth. It is the hardest substance in the body. It has a porous structure made up of 87 volume % of hydroxyapatite crystals (a mineral made of calcium phosphate), 2 volume % organic solids and 11 volume % water. (Some recent studies suggest that the mineral content may be as high as 95%.) The enamel thickness varies over the crown and for the different types of teeth. For the present analysis, an average thickness of 1.6 mm will be assumed. (Underneath the enamel is the dentin with 48% minerals, 28% organics and 24% water. The dentine has the largest volume in the teeth and it protects nerve endings and blood capillaries.) Because of its porosity, coloured chemicals such as tannins, polyphenols and other pigments from tobacco, coffee, red wine and blueberry amongst others penetrate and stain the enamel below the surface. Teeth whitening is done by using gels containing a bleaching agent, hydrogen peroxide (H_2O_2), which releases active oxygen radicals required to breakup the coloured chemicals. Liquid H_2O_2 is seldom used. Carbamide peroxide, a compound of urea and H_2O_2 with 33% H_2O_2 by weight also used for bleaching hair, is mostly applied.

For a procedure professionally done by a dentist, a gel containing H_2O_2 which yields 20 mol % oxygen radical (O°) in water (on dissociation) was applied to cover the teeth. The radical diffused into the enamel (assumed flat, 1.6 mm thick and with 11 volume % water) and, after 15 minutes, just reached the inner surface of the enamel. Assume the water in the enamel is not mobile.

(a) Derive an expression for the concentration of the oxygen radical in the enamel as a function of time in the first 15 minutes. You may assume that the concentration at the outer surface is constant and neglect any chemical reactions. ***Use the integral method.***

(b) Estimate the diffusivity (D_{AB}) of the radical in the water within the enamel.

Data:
The total concentration of the mixture of the radical and water is 55.6 kmols/m^3.

Problem #3 (30 points)

The bottom product of atmospheric distillation columns for crude oil is usually tar which, when it cools down, becomes very viscous and difficult to pump. At a refinery near a big river, the tar is to be pumped to a barge and transported to another plant for cracking and further processing. It was decided that the steel transfer pipe from the refinery to the dock be kept warm by passing an electric current through its uninsulated wall. The rate of heat generation per unit volume is given as g+. The outside radius of the pipe is R and the inside radius is κR. It is assumed that inside the pipe, the velocity profile is fully developed. The internal flow condition is also such that there is not net heat exchange between the tar and the pipe wall at any location, i.e. the tar temperature remained constant from the pipe inlet to the outlet. Outside the pipe, the tempeature of air is constant at T_∞ and the heat transfer coefficient on the outside is h_o, a constant.

Derive an expression for the radial temperature profile through the pipe wall. Show all your steps.

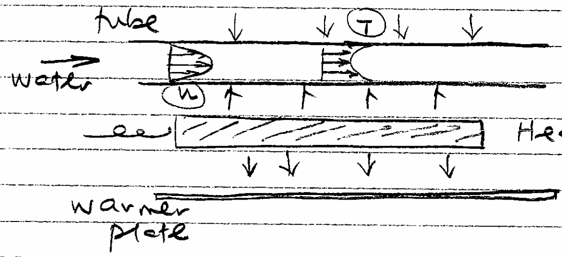
Problem #4 (20 points)

(a) Data for the terminal velocity **U** of a spherical particle of diameter **D** falling through a fluid of density ρ and viscosity μ, under a net "force" relating gravity effect on mass to buoyancy $\mathbf{g(\rho_s - \rho)}$, are to be correlated. Determine the dimensionless groups.

(b) Maple syrup is made by concentrating (evaporating off water from) the sap tapped from maple trees. Approximately 40 litres of sap yields 1 litre of syrup. If the sugar content of the sap is 3% by weight, estimate the sugar content of the syrup. State your assumptions.

ENCH 501 Final Exam Solution - Dec. 16, 2004

#1 This is a problem involving heat transfer into a liquid in laminar flow through a tube. For the analysis, it will be assumed that the tube is straight.



In general, the energy provided by the heater is split between heating up the water in the pipe and keeping the warmer plate hot.

For the minimum time to brew 1 litre of coffee, all the heat should go into heating the water.

In this case, the heat gain by water = Q

or $\dot{m} C_p (T_{out} - T_{in}) = \rho \dot{\forall} C_p (45-15) = Q$

$$\therefore \underset{\text{volume rate}}{\dot{\forall}} = \frac{500}{(995.6)(4186)(30)} = 3.9991(10^{-6})\ m^3/s$$

(a) To brew 1 litre or $10^{-3}\ m^3$,

$$t = \frac{10^{-3}}{3.9991(10^{-6})} = 250.05\ s$$

(b) The wall and centre-line temperatures may be derived from expressions in the Notes, p. 169

$$T_{bm} = T_o + \frac{7}{96} \frac{U_{max} R^2}{\alpha} \frac{dT_{bm}}{dx}$$

$$T_w = T_o + \frac{3}{16} \frac{U_{max} R^2}{\alpha} \frac{dT_{bm}}{dx}$$

$$\therefore T_w - T_{bm} = \frac{11}{96} \left(\frac{U_{max} R^2}{\alpha} \frac{dT_{bm}}{dx} \right)$$

For the problem, $T_{bm}\big|_{inlet} = 15°C$

$T_{bm}\big|_{outlet} = 45°C$

and $L = 0.4\ m$

$$\therefore \frac{dT_{bm}}{dx} = \text{constant} = \frac{30}{0.4} = 75\ °C/m$$

Also $T_{bm}(x) = 15°C + 75x \; ; \; 0 \le x < 0.4$

Since the flow rate of water $= 3.9991(10^{-6})\ m^3/s$

and the x-sectional area $= \dfrac{\pi D^2}{4} = \dfrac{\pi (9.525)^2 (10^{-6})}{4}\ m^2$

(i.d. = 3/8" or 9.525 mm)

$$\bar{u} = \frac{4 \times 3.9991(10^{-6})}{\pi (9.525)^2 (10^{-6})} = 0.0561\ m/s$$

Check Reynolds #

$$Re = \frac{D\bar{u}\rho}{\mu} = \frac{(9.525)(10^{-3})(0.0561)(995.6)}{0.8(10^{-3})}$$

$$= 665.2$$

$\therefore$ Flow is laminar

$\therefore\ U_{max} = 2\bar{u} = 0.1122\ m/s$

$$\alpha = \frac{k}{\rho c_p} = \frac{0.615}{(995.6)(4186)} = 1.4757(10^{-7})\ m^2/s$$

$$\therefore T_w - T_{bm} = \frac{11}{96}\left\{\frac{0.11224(9.525)^2(10^{-6})}{1.4757(10^{-7})\,4} \times 75\right\}$$

$$= 148.25\ °C$$

A $x = L/2 = 0.2\,m$, $T_{bm} = 30°C$

$\therefore\ T_w(x = 0.2\,m) = 178.25\ °C \longrightarrow$

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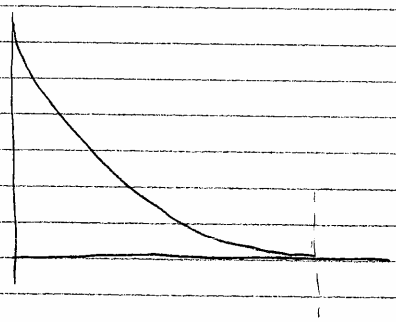

Treat the enamel as a semi-infinite body for the first 15 minutes.

Species balance for O:

$$N_A\Big|_{x=0} = \frac{d}{dt}\int_0^{\delta_m} c\,\varepsilon\, y_A\, dx \quad (1)$$

This is Eq. 5.91, Notes p. 120.

Note that ε is the porosity of medium, or the fraction occupied by water in the enamel.

The Flux at $x=0$ is derived from

$$N_A = -c D_{AB}\varepsilon \frac{dy_A}{dx} + y_A(N_A + \underbrace{N_B}_{0})$$

water immobile.

$$\therefore\ N_A = -\frac{c\varepsilon D_{AB}}{1-y_A}\frac{dy_A}{dx}$$

By Integral method, identify b.c.

$$x=0 \qquad y_A = y_{A0}$$
$$x=\delta_m \qquad y_A = 0$$
$$x=\delta_m \qquad \frac{dy_A}{dx} = 0$$

$\therefore$ Assume $y_A = a + bx + cx^2$

Use b.c.

$$\frac{y_A}{y_{A0}} = \left(1-\frac{x}{\delta_m}\right)^2 \quad (2)$$

Substitute profile into balance equation

$$-\left(\frac{c\,\varepsilon D_{AB}}{1-y_A}\frac{dy_A}{dx}\right)\Bigg|_{x=0} = \frac{d}{dt}\left[y_{A0}\,\varepsilon\, c\,\frac{\delta_m}{3}\right]$$

$$\frac{dy_A}{dx} = \frac{-2y_{A0}}{\delta_m}\left(1-\frac{x}{\delta_m}\right) \quad \text{or} \quad \left.\frac{dy_A}{dx}\right|_{x=0} = -\frac{2y_{A0}}{\delta_m}$$

$$\left.\frac{1}{1-y_A}\right|_{x=0} = \frac{1}{1-y_{A0}\left(1-\frac{x}{\delta_m}\right)^2} = \frac{1}{1-y_{A0}}$$

$$\therefore \quad \frac{2c\,\varepsilon\, D_{AB}\, y_{A0}}{\delta_m\,(1-y_{A0})} = \frac{y_{A0}\, c\,\varepsilon}{3}\,\frac{d\delta_m}{dt}$$

$$\frac{6D_{AB}}{1-y_{A0}} = \delta_m\frac{d\delta_m}{dt} = \frac{1}{2}\frac{d\delta_m^2}{dt}$$

Subject to the condition — $t=0$, $\delta_m=0$

$$\delta_m = 2\sqrt{\frac{3D_{AB}t}{1-y_{A0}}}$$

(a) Substitute this into eq. (2) for profile. →

(b) $\delta_m = 1.6(10^{-3})\ m$

$y_{A0} = 0.2$

$t = 15(60) = 900\ s$

$$\therefore \quad D_{AB} = 1.8963\,(10^{-10})\ m^2/s$$

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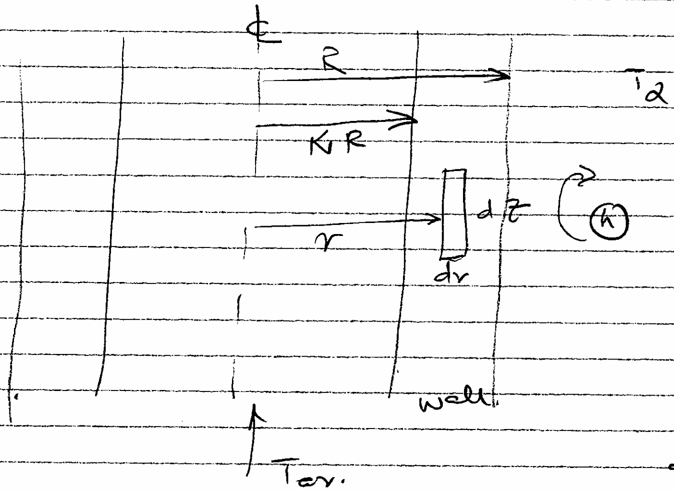
#3 This problem is similar to 'conduction in solids with internal heat generation' on p. 163 Notes.



Write an energy balance on the differential element, a ring (as shown in sketch).

Input + Gen = Output + Accum.

$$2\pi r\,dz\,q_r\Big|_r + g^+(2\pi r\,dr\,dz)$$

$$= 2\pi r\,dz\,q_r\Big|_{r+dr} + \underbrace{\text{Accum}}_{\to 0 \text{ steady state}}$$

Hence equation becomes

$$\frac{1}{r}\frac{d(r q_r)}{dr} = g^+ \qquad (1)$$

Integrate once, $r q_r = \dfrac{g^+ r^2}{2} + C_1$

or $$-k\frac{dT}{dr} = \frac{g^+ r}{2} + \frac{C_1}{r} \qquad (2)$$

use b.c. at $r = kR$ (inner surface)

$$\frac{dT}{dr} = 0$$

$$\Rightarrow \quad C_1 = -\frac{g^+(kR)^2}{2} \qquad (3)$$

Integrate (2)

$$T = -\frac{g^+}{4k}r^2 - \frac{C_1}{k}\ln r + C_2 \qquad (4)$$

with b.c. $r = R$, $-k\frac{dT}{dr} = h(T - T_\alpha)$

Also, since all the heat generated leaves at the outer surface,

$$h(\cancel{2\pi} R \cancel{L})(T - T_\alpha) = g^+(\cancel{\pi}(1-\kappa^2)R^{\cancel{2}})\cancel{L}$$

$$\text{or} \quad T = T_\alpha + \frac{g^+(1-\kappa^2)R}{2h}$$

One can use this condition at $r = R$

Hence

$$T_\alpha + \frac{g^+(1-\kappa^2)R}{2h} = -\frac{g^+R^2}{4k} + \frac{g^+(\kappa R)^2}{2k}\ln R + C_2$$

$$T_\alpha + \frac{g^+(1-\kappa^2)R}{2h} + \frac{g^+R^2}{2k}\left[\frac{1}{2} - \kappa^2\ln R\right] = C_2 \qquad (5)$$

The temperature profile in the wall is given by combining (3), (4) and (5).

#4 (a)

$u = f(D, \rho, \mu, g(\rho_s - \rho))$ # variables = 5

units: m/s, m, kg/m^3, $Pa.s$ or $N/m^2 . s$, $m/s^2 . kg/m^3$

Dimensions: $\frac{L}{t}$, L, $\frac{M}{L^3}$, $\frac{M}{Lt}$, $\frac{M}{L^2t^2}$ # of dimensions = 3

# of Dimensionless variables — 5 - 3 = 2

$\Pi_1 = u^a D^b \mu^c \rho$ and $\Pi_2 = u^a D^b \mu^c g(\rho_s - \rho)$

$0 = \left[\frac{L}{t}\right]^a \left[L\right]^b \left[\frac{M}{Lt}\right]^c \left[\frac{M}{L^3}\right]$ | $0 = \left[\frac{L}{t}\right]^a \left[L\right]^b \left[\frac{M}{Lt}\right]^c \left[\frac{M}{L^2t^2}\right]$

mass: $0 = c + 1$ | $0 = c + 1$

Length: $0 = a + b - c - 3$ | $0 = a + b - c - 2$

time: $0 = -a - c$ | $0 = -a - c - 2$

$c = -1$, $a = 1$, $b = 1$ | $c = -1$, $a = -1$, $b = 2$

$\Pi_1 = \frac{uD\rho}{\mu}$, Reynolds number | $\Pi_2 = \frac{D^2 g(\rho_s - \rho)}{u\mu}$

(b) Basis: 100 kg sap — 97% water

Assume $\rho_{water} = 1000\ kg/m^3$

Vol. water $= 0.097 m^3$ initial $+ \frac{0.097}{40}$ final

∴ Mass of water final $= 2.425$

∴ In syrup, conc. $= \frac{\text{mass sugar}}{\text{mass sugar + water}} = \frac{3}{2.425 + 3}$

∴ % sugar $= 55.3\ \%$

→