G. J./pm.

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Department of Chemical & Petroleum Engineering

ENCH 501: Mathematical Methods in Chemical Engineering

Final Examination, Fall 2002

Time: 8 - 11 am **Wednesday, December 11, 2002**

Instructions: **Attempt All Questions.**
Use of Electronic Calculators allowed.
Open Notes, Open Book Examination.

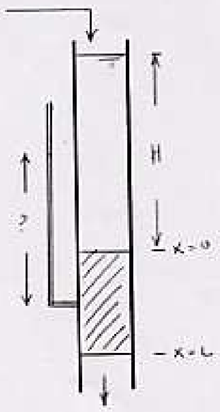
Problem #1 (25 points)

At least once a year, the water in the Glenmore reservoir undergoes a thermal inversion, i.e. heavy cold water which accumulated above lighter warm water sinks and convection currents stir up the mud at the bottom of the lake. Water which then enters the intake duct of the distribution pipeline (for the period before the mud has settled) contains suspended clay particles and it is brownish. The water also contains traces of dissolved and suspended organic substances and is cleaned or purified by passing it through a bed of activated charcoal particles.

A bed of particles or a filter is as shown in the sketch below. The bottom end is exposed to the ambient. The cross -section of the bed is rectangular and is 50 by 60 cm. The height (L) of the bed is 80 cm and a column of water (H) 1.2 m high is maintained above the bed. The clay particles suspended in water deposited within the bed and changed the permeability non-uniformly. Starting from the top of the bed (x=0) downwards, the permeability is described by:

$$\kappa = \kappa_o(1 - a e^{-b\eta}) \qquad \text{where } \eta = x/L,\ a = 0.7389 \text{ and } b=2$$

and κ_o equal 5 darcys.



(a) How long would be required for 500 US gallons (1 US gallon = 3.78 litres) of clear water to percolate through the bed at steady state?

(b) If an L-shaped tube is attached at a point mid-way of the bed, what would be the level of water in the side arm above the point of attachment?

Data:
Properties of water: μ=1.308 mPa s, ρ=999.8 kg/m^3

Darcy's Law: $Q = (\kappa/\mu)A[-dP/dx + \rho g]$

1 darcy = $9.78(10^{-13})$ m^2

Problem #2 (30 points)

Public swimming pools and other recreational water venues need to be disinfected in cases of fecal accidents. It is recommended that hypochlorites be added and the concentration of free chlorine be raised to 20 mg/litre (or 20 ppm) at a pH of 7.2 to 7.5 for at least 8 hours to ensure that *Cryptosporidium* present is inactivated. People who swim in the water soon after disinfection absorb a considerable amount of chlorine which is "tough" on the skin. You have been assigned the task of estimating the uptake.

Available to you are 10 cm long and 1.2 cm diameter tubes filled with a gelatin-like material and sealed at one end. The diffusivity of the medium at $5(10^{-9})$ m^2/s is similar to that of the outer layers of the skin when wet. The convective mass transfer coefficient (k_L) at the gelatin-water surface is 0.0027 m/s and the free chlorine concentration in the pool was maintained at 20 ppm. The gelatin (density ~1020 kg/m³) initially did not contain any chlorine.

(a) Use the ***Integral method*** to derive an expression relating the depth to which chlorine would have penetrated into the gelatin (δ) and time elapsed (t) from the instant of tube immersion in the pool. Make reasonable assumptions and clearly justify them.

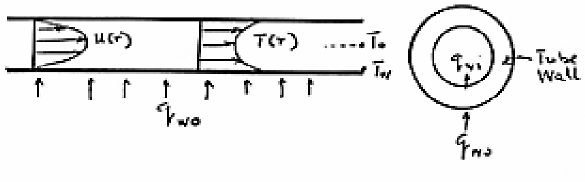
(b) Determine how much chlorine would have been absorbed by the gelatin after 8 hours of immersion of a tube.

Problem #3 (20 points)

Aviation fuel is to be pre-heated before being injected into an aircraft engine. The fuel flows through a ½ inch Schedule 80XS steel tube wrapped with an electric heating tape and insulated on the outer surface. The fluid enters the heating zone with the velocity fully developed and at a temperature of 15°C. It is assumed that the heat flux is constant along the length of the tube and the fuel is to exit the tube at a bulk mean temperature of 75°C. You may assume the temperature profile is very rapidly developed at the start of the heating zone and the tube length is 2 m. The fluid flow is laminar at a Reynolds number of 190.

If you can neglect the thermal resistance of the tube wall, estimate the ***heat flux*** at the outer surface of the wall of the tube (q_W), the ***centre-line*** (T_o) and the ***tube wall*** (T_W) temperatures at the pipe outlet.

Data: Properties of aviation fuel:
ρ = 750 kg/m³ ; μ = 0.586 mPa s ; C_p = 2.132 kJ/kg K ; k = 0.136 W/m K



Problem #4 (20 points)

Electricity is passed axially along a hollow aluminum pipe of outer radius 4 cm and a wall thickness of 1 cm. Air is blown over the outer surface and through the pipe at high rates such that the surface temperatures are maintained at 50°C while the steady state maximum temperature within the wall is given as 120°C.

At what ***radial distance*** is the maximum temperature located and what is the rate of ***heat generation*** per unit volume in the metal? You may assume that the current density is such that the rate of heat generation is constant throughout the solid.

Data: Properties of aluminum:
ρ = 2707 kg/m^3 ; C_p = 0.869 kJ/kg K ; k = 206 W/m K

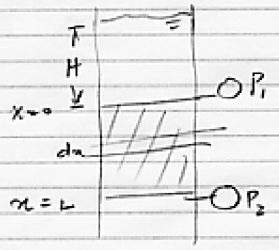
Problem #5 (BONUS 5 points)

Sketch the following. Give a brief description of how they work and typical applications:

- globe valve
- check valve
- centrifugal pump
- a heat pipe
- a cooling tower

ENCH 501 Solution - FINAL EXAM Dec. 11, 2002

☐ Problem # 1 Flow through porous media



For the vertical orientation, Darcy's law is written as

$$Q = \frac{k_r A}{\mu}\left[-\frac{dP}{dx} + \rho g\right] = vA$$

where $k_r = k_r(x)$

Perform a material balance on the differential element dx, at steady state

$$\frac{d(\rho v A)}{dx}\,dx = 0$$ where superficial velocity v is defined above.

Hence $$\frac{d}{dx}\left[A \rho \frac{k_r}{\mu}\left(-\frac{dP}{dx} + \rho g\right)\right] = 0$$

or $$\frac{\rho A}{\mu} \frac{d}{dx}\left[k_r\left(-\frac{dP}{dx} + \rho g\right)\right] = 0$$

Integrate once

$$k_r\left(-\frac{dP}{dx} + \rho g\right) = C_1$$

substitute definition for k_r, Let $\eta = x/L$

$$k_o\left(1 - a e^{-b\eta}\right)\left[-\frac{dP}{L\,d\eta} + \rho g\right] = C_1$$

$$\therefore \frac{dP}{d\eta} = \rho g L - \frac{C_1 L}{k_o\left(1 - a e^{-b\eta}\right)}$$

Integrate $$P = \rho g L \eta - \frac{L C_1}{k_o}\int \frac{d\eta}{\left(1 - a e^{-b\eta}\right)} + C_2$$

From tables of indefinite integrals

$$P = \rho g L\eta - \frac{LC_1}{K_0}\left[\eta + \frac{1}{2}\ln(1 - ae^{-2\eta})\right] + C_2$$

Use b.c. $x = 0 = \eta \quad P_1 = H\rho g'$ (gauge) ①

$x = L$ or $\eta = 1 \quad P_2 = 0$ ②

Use b.c. ①

$$H\rho g' = -\frac{LC_1}{K_0}\left[\frac{1}{2}\ln(1-a)\right] + C_2$$

Use b.c. ②

$$0 = \rho g L - \frac{LC_1}{K_0}\left[1 + \frac{1}{2}\ln(1 - ae^{-2})\right] + C_2$$

subtract

$$0 = (H+L)\rho g - \frac{LC_1}{K_0}\left[1 + \frac{1}{2}\ln(1 - ae^{-2}) - \frac{1}{2}\ln(1-a)\right]$$

or $$\boxed{C_1 = \left(\frac{K_0}{L}\right)\frac{(H+L)\rho g}{1 + \frac{1}{2}\ln\left(\frac{1-ae^{-2}}{1-a}\right)}}$$

$$= \frac{5(9.87)(10^{-13})}{0.8}\;\frac{(1.2+0.8)(999.8)(9.81)}{1 + \frac{1}{2}\ln\left(\frac{1-0.7389e^{-2}}{1-0.7389}\right)} = 7.4753(10^{-8}) \text{ Pa.m}$$

But percolation rate $Q = \frac{A}{\mu}C_1$

$A = (0.5)(0.6)\,m^2$, $\mu = 1.308(10^{-3})$ Pa.s.

$\therefore \quad Q = 1.7145(10^{-5})\ m^3/s$

Total volume $\forall = 500$ US gal $= 1.89\ m^3$

(a) Time $t = \frac{\forall}{Q} = 110{,}234.5\,s$ or 30.62 hours

c_2 is determined from

$$c_2 = H\rho_s g + \frac{L c_1}{K_{ro}}\left[\frac{1}{2}\ln(1-a)\right] = 3633.3 \text{ Pa}$$

Hence the relation $P(\eta)$ or $P(x/L)$ is

$$P = \rho g L \eta - \frac{L c_1}{K_{ro}}\left[\eta + \frac{1}{2}\ln\left(1 - a e^{-2\eta}\right)\right] + 3633.3$$

(b)

When $\eta = \frac{1}{2}$

$$P = (999.8)(9.81)(0.8)(0.5) - \frac{0.8(7.4753)(10^{-8})}{5(9.87)(10^{-13})}\left[\frac{1}{2} + \frac{1}{2}\ln\left(1 - 0.7389\, e^{-1}\right)\right] + 3633.3$$

$$= 3419.52 \text{ Pa} = h\rho_s g$$

$\therefore$ Height of liquid $h = \frac{3419.52}{999.8(9.81)} = 0.3486$ m

or 34.9 cm $\longrightarrow$

ALTERNATE SOLUTION for Problem #1

(a) Given

$$Q = \frac{\kappa}{\mu} A \left[-\frac{dP}{dx} + \rho g' \right]$$

At steady state, Q = constant

(i) $$\frac{dP}{dx} = -\frac{\beta}{\kappa} + \rho g' \quad ; \quad \beta = \frac{\mu}{A} Q$$

$$\kappa = \kappa_0 (1 - a e^{-b\eta})$$

$$\frac{1}{L} \frac{dP}{d\eta} = -\frac{\beta}{\kappa_0 (1 - a e^{-b\eta})} + \rho g'$$

(ii) $$\int_{H\rho g'}^{0} dP = \gamma \int_0^1 \frac{d\eta}{1 - a e^{-b\eta}} + \rho g' L \int_0^1 d\eta \quad ; \quad \gamma = -\frac{\beta L}{\kappa_0}$$

$$-H\rho g' = \gamma \left[\eta + \frac{1}{2} \ln \left(1 - a e^{-2\eta} \right) \right]_0^1 + \rho g' L$$

$$= \gamma \left[1 + \frac{1}{2} \ln \left(\frac{1 - 0.7389 e^{-2}}{1 - 0.7389} \right) \right] + \rho g' L$$

$$-(L + H)\rho g' = -\frac{\mu Q}{A} \frac{L}{\kappa_0} \left[1.6187 \right]$$

(iii) $$\therefore \quad Q = \frac{(L + H) \rho g' \cdot A \kappa_0}{\mu L (1.6187)}$$

$$= \frac{2 (999.8)(9.81)(0.05)(5)(9.87)(10^{-13})}{1.308 (10^{-3})(0.8)(1.6187)}$$

$$= 1.7145 (10^{-5}) \ m^3/s$$

$$\therefore \quad \text{time} = t = V/Q = \frac{1.89 \ m^3}{Q} = 1.1023 (10^5) \ s \longrightarrow$$

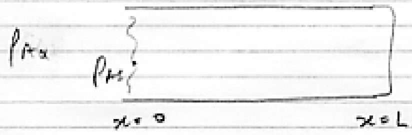
(b) For this part, use eq. (ii) with the upper limit for the L.H.S. set at $x\rho g'$ (instead of zero) where x is the height of liquid in the side arm above the attachment, and set the upper limit for the r.h.s. at $\eta = \frac{1}{2}$. With Q known from part (a), x can be evaluated from

$$(x - H)\rho g' = \gamma\left[\eta + \tfrac{1}{2}\ln\left(1 - a e^{-2\eta}\right)\right]_0^{\frac{1}{2}} + \rho g'\frac{L}{2} \qquad \text{(iv)}$$

$x = 0.3486$ m.

$\longrightarrow$

[1] Problem #2 Diffusion in a semi-infinite medium: Integral method.



Material Balance on Chlorine

- No reactions ; Mass Balance

Input + Gen = Output + Accum.

$$\text{Gen} \to 0, \quad \text{Output} \to 0$$

By the integral method ① $$n_A\Big|_{x=0} = \frac{d}{dt}\left[\int_0^L \rho_A \, dx\right]$$ analogous to eq. 5.46 Notes

where $$n_A = -\rho D_{AB}\frac{\partial w_A}{\partial x} + w_A(n_A + n_B)$$

- Since w_A is small, neglect convective term, i.e. $w_A(n_A+n_B)$

Hence $$n_A = -\rho D_{AB}\frac{\partial w_A}{\partial x} = k_L(\rho_{A\infty} - \rho_{AS})$$

- where ρ = constant and $\rho_{AS}(t)$; $\rho_{A\infty}$ = const.

$\therefore$ a Boundary condition is at $x=0$

② $$n_A\Big|_{x=0} = -D_{AB}\frac{\partial \rho_A}{\partial x} = k_L(\rho_{A\infty} - \rho_{AS})$$ similar to eq. 5.60 Notes

Other conditions are

③+④ $x = \delta(t)$ $\quad \rho_A = 0$ and $\dfrac{d\rho_A}{dx} = 0$

Assume a concentration profile of the type (in the gelatin)

$$\frac{\rho_A}{\rho_{AS}} = a + b\left(\frac{x}{\delta}\right) + c\left(\frac{x}{\delta}\right)^2$$

B.Cs ③ + ④ with $\rho_A = \rho_{AS}$ at $x=0$ yeilds

⑤ $$\frac{\rho_A}{\rho_{AS}} = \left(1 - \frac{x}{\delta}\right)^2$$

⑥ $$\left.\frac{d\rho_A}{dx}\right|_{x=0} = \left. 2\rho_{AS}\left(-\frac{1}{\delta}\right)\left(1-\frac{x}{\delta}\right)\right|_{x=0} = -\frac{2\rho_{AS}}{\delta}$$

On combination with ②

⑦ $$\left.\frac{d\rho_A}{dx}\right|_{x=0} = -\frac{k_L}{D_{AB}}(\rho_{A\alpha} - \rho_{AS}) = -\frac{2\rho_{AS}}{\delta}$$

Substitute ⑤ and ② into ①

$$k_L(\rho_{A\alpha} - \rho_{AS}) = \frac{d}{dt}\left[\int_0^\delta \rho_{AS}\left(1-\frac{x}{\delta}\right)^2 dx\right]$$

$$= \frac{d}{dt}\left[\rho_{AS}\,\delta \int_0^1 (1-\eta)^2\, d\eta\right]$$

⑧ $$k_L(\rho_{A\alpha} - \rho_{AS}) = \frac{d}{dt}\left[\rho_{AS}\,\delta/3\right]$$ with 2 dependent variables ρ_{AS} + δ

or using ⑦

$$\frac{2\, D_{AB}\,\rho_{AS}}{\delta} = \frac{d}{dt}\left[\rho_{AS}\frac{\delta}{3}\right]$$

Also from ⑦ $$\rho_{AS} = \frac{\beta\rho_{A\alpha}}{\beta + 2/\delta}$$ where $\beta = \frac{k_L}{D_{AB}}$

or

⑨ $$\frac{6\, D_{AB}}{\beta\delta + 2} = \frac{d}{dt}\left[\frac{\delta^2}{\beta\delta+2}\right]$$ subject to $t=0,\ \delta=0$.

or $$\frac{6\, D_{AB}}{\cancel{\beta\delta+2}} = \frac{1}{\cancel{\beta\delta+2}}\frac{d\delta^2}{dt} - \frac{\delta^2}{(\beta\delta+2)^{\cancel{2}}}\frac{d\delta}{dt}\cdot\beta$$

$$6\, D_{AB} = \frac{d\delta^2}{dt} - \frac{\beta\delta^2}{\beta\delta+2}\frac{d\delta}{dt}$$

Integrate

$$6 D_{AB}\, t = \delta^2 - \int_0^\delta \frac{\beta \delta^2}{\beta\delta + 2}\, d\delta$$

(a)

$$6 D_{AB}\, t = \frac{\delta^2}{2} + \frac{2\delta}{\beta} - \frac{4}{\beta^2} \ln\left[1 + \frac{\beta\delta}{2}\right]$$ Q.E.D.

(b) Require δ at $t = 8$ hrs — solve $\delta = 0.041566$ m

Total Cl_2 absorbed $Q = A \int_0^\delta \rho_A \, dx$

$$= A \int_0^\delta \rho_{As} \left(1 - \frac{x}{\delta}\right)^2 dx$$

$$Q = A \rho_{As}\, \delta \int_0^1 (1-\eta)^2 d\eta$$

$$= A \left(\frac{\beta \rho_{A\infty}}{\beta + 2/\delta}\right) \frac{\delta}{3}$$

$$\beta = \frac{k_c}{D_{AB}} = \frac{2.7(10^{-3})}{5(10^{-9})} = 5.4(10^5)$$

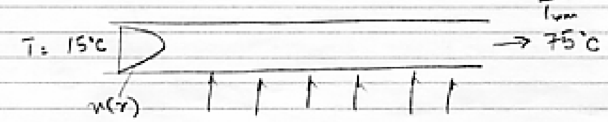
$\rho_{A\infty} = 20(10^{-3})$ kg/m^3 ; $A = (0.5)(0.6) = 0.3$ m^2

$$Q = 0.5 \left[\frac{5.4(10^5)\, 20(10^{-3})}{5.4(10^5) + \frac{2}{0.041566}}\right] \frac{0.041566}{3} \text{ kg}$$

$$= 8.3125(10^{-5}) \text{ kg} \quad \text{or} \quad 83.125 \text{ mg}$$

☐ Problem # 3

Notes: p 167 – 169



steps

Calc. $\dot{m} C_p \Delta T$

Calc. $q_{wo} = \dfrac{\dot{m} C_p \Delta T}{\text{Area}}$

obtain $\dfrac{dT_{bm}}{dx}$ – from q_{wi}

solve for T_0, T_w

$$Re = \frac{D\bar{u}\rho}{\mu} = 1900$$

$\frac{1}{2}$" schedule 80XS tube

⇒ o.d. = 0.84" or 2.1336 cm
i.d. = 0.546" or 1.3868 cm

$$\therefore \quad \frac{1.3868(10^{-2})\,\bar{u}\,(750)}{0.586(10^{-3})} = 190$$

or $\bar{u} = 0.01070$ m/s $\qquad A = \dfrac{\pi D^2}{4}$, D = i.d.

$$\dot{m} = \bar{u} A \rho = 0.01070\left(\frac{\pi}{4}\right)(1.3868)^2(10^{-4})\,750 = 0.0012131\ \text{kg/s}$$

Rate of heat gain by fuel = $\dot{m} C_p \Delta T_{bm} = 0.00121(2132)(75-15)$

$$Q = 155.13\ \text{J/s}$$

∴ Heat flux $= \dfrac{Q}{\pi D_o L}$ where D_o is outside diameter

Heat flux on outside surface

$$q_{wo} = \frac{155.12}{\pi(2.1336)(10^{-2})\,2} = 1{,}157.179\ \text{W/m}^2$$

⟶

from equation 6.97 Notes

$$\frac{dT_{bm}}{dx} = \frac{2 q_{wi}}{R \bar{u} \rho C_p}$$

where $R = \dfrac{\text{i.d.}}{2}$ and $q_{wi} = q_{wo}(D_o/D_i)$

$$= \frac{2(1157.179)\,2}{(1.3868)(10^{-2})(0.0107)(750)(2132)} = 30.01\ \frac{°C}{m}$$

This is consistent since the temp. rise was 60°C in 2m length.

Since $T_{bm} = T_0 + \frac{7}{96} \frac{u_{max} R^2}{\alpha} \frac{dT_{bm}}{dx}$

At the outlet of pipe, $T_{bm} = 75°C$

$u_{max} = 2\bar{u} = 0.0214 \text{ m/s}$

$\alpha = k/\rho C_p = \frac{0.134}{750(2137)} = 8.5053 (10^{-8})$

$R = \frac{1.3868(10^{-2})}{2}$ m

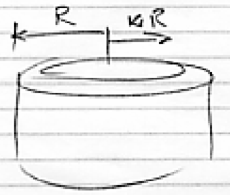
$\therefore \quad T_0 = 75 - \frac{7}{96} \frac{(0.0214)(1.3868)^2(10^{-4})}{4\,(8.5053)(10^{-8})} (30.01)$

At exit, centreline temp. $T_0 = 75 - 26.474 = 48.53°C$ →

Wall temp.
$T_w = 48.53 + \frac{3}{16} \frac{(0.0214)(1.3868)^2(10^{-4})}{4\,(8.5053)(10^{-8})} (30.01)$

At exit
$T_w = 48.53 + 68.05 = 116.57°C$ →

1) Problem #4



Notes – p. 163-164

$\kappa = 0.75$

The energy equation is eq. 6.87

$$\frac{1}{r}\frac{d}{dr}(r q_r) = \phi \quad , \quad \text{rate of heat generation per unit volume}$$

and the integration yeilds

$$① \quad T = -\frac{\phi}{4k} r^2 - \frac{c_1}{k}\ln r + c_2$$

where the constants c_1 & c_2 are found from the b.c. $r = \kappa R$ and $r = R$, $T = T_o$

$$\therefore \quad c_1 = -\frac{\phi}{4} \frac{R^2(1-\kappa^2)}{\ln(1/\kappa)}$$

$$c_2 = T_o + \frac{\phi (\kappa R)^2}{4k} - \phi \frac{\ln(\kappa R)}{4k} \cdot \frac{R^2(1-\kappa^2)}{\ln(1/\kappa)}$$

The heat flux (radial) is

$$q_r = \frac{\phi r}{2} - \frac{\phi}{4r} \frac{R^2(1-\kappa^2)}{\ln(1/\kappa)} \qquad \text{eq } 6.89$$

ⓐ At location of max. temp, $q_r = 0$

$$\therefore \quad \frac{\cancel{\phi} r^2}{\cancel{2}} = \frac{\cancel{\phi}}{\cancel{4}\, 2} \frac{R^2(1-\kappa^2)}{\ln(1/\kappa)}$$

or $$r = R\left[\frac{1-\kappa^2}{2\ln(1/\kappa)}\right]^{\frac{1}{2}} = 0.872\,R$$

ⓑ ϕ may be determined from the maximum temperature.

$$T = -\frac{\phi}{4k} r^2 - \frac{c_1}{k} \ln r + c_2$$

$r = \kappa R \qquad T = T_o$

$r = R \qquad T = T_o$

$$T_o = -\frac{\phi}{4k} (\kappa R)^2 - \frac{c_1}{k} \ln \kappa R + c_2$$

$$T_o = -\frac{\phi}{4k} R^2 - \frac{c_1}{k} \ln R + c_2 \qquad \kappa = 0.75$$

$$0 = \frac{\phi}{4k} R^2 (1-\kappa^2) + \frac{c_1}{k} \ln \frac{R}{\kappa R}$$

$$\therefore \quad C_1 = -\cancel{k} \frac{\phi R^2}{4\cancel{k}} \frac{(1-\kappa^2)}{\ln(1/\kappa)} = \frac{\phi R^2}{4} (-1.5208)$$

$$C_2 = T_o + \frac{\phi}{4k} R^2 + \frac{\phi R^2 (1-\kappa^2)}{4k \ln \kappa} \ln R$$

$$= T_o + \phi \left[\frac{R^2}{4k} + \frac{R^2 (1-\kappa^2)}{4k \ln \kappa} \ln R \right]$$

$$C_2 = T_o + \phi \frac{R^2}{4k} \left[1 + \frac{1-\kappa^2}{\ln \kappa} \ln R \right]$$

$$= T_o + \frac{\phi R^2}{4k} (5.8952)$$

$$120 = -\frac{\phi}{4k}(\lambda R)^2 - (-1.5208)\frac{\phi R^2}{4k}\ln(\lambda R) +$$

$$T_o + \frac{\phi R^2}{4k}(5.8952)$$

50

$$70 = \frac{\phi R^2}{4k}\left[-\lambda^2 + 1.5208\ln(\lambda R) + 5.8952\right]$$

$$\lambda = 0.872$$

$$70 = \frac{\phi R^2}{4k}\left[5.8952 - 5.8639\right]$$

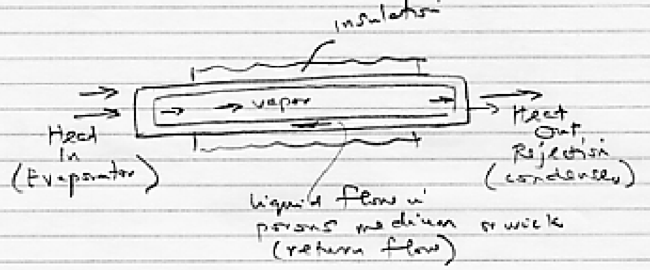
$$= \frac{\phi R^2}{4k}\left[0.0313\right]$$

$$\phi = \frac{70}{0.0313} \cdot \frac{4\,(206)}{(0.04)^2} \quad W/m^3$$

$$= 1.1535\,(10^9)\ W/m^3$$

$\longrightarrow$

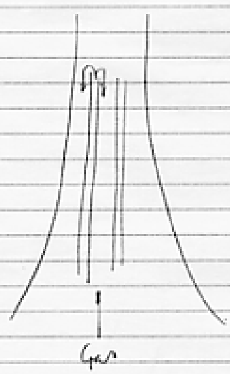
(d) Heat Pipe.



- Device to transfer heat. More efficient than a copper rod of similar dimensions.
- Material for wick, shell + fluid may vary with intended application.

Applications: Heat loss from small spaces - e.g. microelectronic devices.

(e) Cooling Towers



- Based on evaporative cooling of water. There are several designs - wetted wall, spray, natural draft;
- Heat transfer is mostly latent. ~ 80%

Applications → cooling of large volumes of water involved with power generation etc.