The University of Calgary
Department of Philosophy

Philosophy 479L01/679.06L01
LOGIC III: GÖDEL's INCOMPLETENESS THEOREMS

Winter 2004 — Richard Zach

Course Outline

Instructor:	**Richard Zach**
Office:	1254 Social Sciences
Office Hours:	TuTh 12:30–1:15 or by appointment
Phone:	220–3170 email: rzach@ucalgary.ca
Lectures:	Tu/Th 11:00–12:15
	129 Science A

Course Description

This course is a continuation of Phil 379 (Logic II). Whereas Logic II concentrates on Turing machines as a model of computation, we will focus on recursive definability in this course. Following a study of recursive functions and sets, including a proof of the equivalence of recursive functions and Turing machines, we will then go on to prove the celebrated Incompleteness Theorems due to Kurt Gödel. These concern first-order theories of arithmetic. The first incompleteness theorem states that no recursive consistent arithmetical theory T is strong enough to decide all the sentences of arithmetic (i.e., there will always be sentences which it neither proves nor disproves). The second incompleteness theorem states that the sentence that says "the theory T is consistent" is such an undecidable sentence. We'll prove these and some related theorems and corollaries.

In the last few weeks of the term, we will look more closely at some of the preceding results and those from Phil 379. These are the metalogical properties and limitations of first-order logic. If we extend first-order logic by quantifiers over subsets of the domain (not just individuals), we get second-order logic. We'll study the expressive power of second-order logic and its limitations (e.g., compactness, completeness, and Löwenheim-Skolem Theorems fail for second-order logic). The status of second-order logic is a hotly debated topic in philosophy of logic and mathematics (Is it really logic, or is it "set theory in sheep's clothing?"), and it is of central importance in computational complexity theory (finite model theory). Another topic we'll look at in more detail is decidability: In Phil 379 you proved that first-order logic is undecidable; and in the first half of the course we prove that arithmetic is undecidable as well. So we'll look at some cases where we do have decidability: monadic logic and Presburger Arithmetic.

Prerequisites and Preparation

Logic II (PHIL 379) is a prerequisite for this course.

It can't hurt to review the material from PHIL 379, especially Chapters 1, 2, 9, 10, 12 of the textbook (Chapters 1, 2, 3, 9, 12, 13 of the 3rd edition).

Required Text

George S. Boolos, John P. Burgess, Richard C. Jeffrey, *Computability and Logic*, **4th edition**, Cambridge University Press

We will cover roughly Chapters 6–8, 15–18, plus some additional material from Chapters 21, 22, 24. Note that the 4th edition contains some significant changes vis-a-vis the 3rd edition. If at all possible, you should get a copy of the 4th edition.

Evaluation

A "diagnostic" homework assignment (5%), 4 homework assignments (40%), an in-class midterm exam (20%), and a take-home final exam (30%). Class participation counts for 5% of your grade. (If you are shy and do not want to speak in class, 4 substantive, serious posts over the course of the term on the online discussion board will earn you an A for participation. Only posts made before the due date of the final exam count. If all your posts are made within a 7-day period, you will receive a maximum credit of 2 grade points for them.) You must submit all four assignments and complete both exams. You must receive a D (average of at least 0.9 points) or better on the final exam to pass the course.

Graduate students must complete 4 homework assignments and a take-home final. Problems will be somewhat harder than those for undergraduate students.

On each problem on an assignment and exam you will receive a letter grade reflecting the level of mastery of the material shown by the work you submit. According to the *Calendar*, letter grades are defined as follows:

A Excellent—superior performance, showing comprehensive understanding of subject matter. (Your solution to an assigned problem shows that you understand the problem and how to solve it; the solution is complete and rigorously correct, and is reasonably direct and elegant.)

B Good—clearly above average performance with knowledge of subject matter generally complete. (You understand the problem and give a complete solution, although there may be minor gaps in the proof, or the solution is correct but circuitous.)

C Satisfactory—basic understanding of the subject matter. (You understand what the question is asking but your solution contains significant errors or gaps.)

D Minimal pass—marginal performance. (You show some knowledge of what is asked, but you don't come near a solution.)

F Fail—Unsatisfactory performance. (It is not clear that you understand what the question is asking, or your proposed solution goes completely in the wrong direction.)

The correspondence of letter grades with grade points is defined in the *Calendar* (A = 4, B = 3, C = 2, D = 1, F = 0). "Slash" grades are possible with grade point values 0.5 below the higher grade (e.g., A/B = 3.5).

In computing your final grade, your marks will be converted to grade points and averaged according to the weights given above. The final grade will be the letter grade corresponding to the weighted average of your assignments, exams, and participation plus a margin of 0.1. For the final grade, +'s and −'s are possible, too; as defined in the *Calendar*, +/− adds/subtracts 0.3 grade points. In other words, a course average of 3.9 or higher receives an A; between 3.6 and 3.9, an A–; between 3.2 and 3.6, a B+; between 2.9 and 3.2, a B; and so on. There is no D− grade; to earn a D you require a course average of at least 0.9. The A+ grade is reserved for "truly outstanding" performance.

Assignments and Policies

Late work and extensions. Assignments handed in late will be penalized by the equivalent of one grade point per calendar day. If you turn an assignment in late, you must give it to me personally or put it in the department drop-box (it will then be date-stamped by department staff). Note that the drop-boxes are cleared at 4 pm, the department closes at 4:30 pm on weekdays and *is closed Saturdays and Sundays*.

There will be no make-up exams under normal circumstances (i.e., unless you can document an illness or other emergency which prevented you from taking the exam); for the final exam, university policies for deferral of exams apply.

Collaboration. Collaboration on homework assignments is encouraged. However, you must write up your own solutions, and obviously you must not simply copy someone else's solutions. You are also required to list the names of the students with whom you've collaborated on the assignment. **If you collaborate without following these instructions, it constitutes cheating.** Of course, no collaboration is allowed on exams.

Plagiarism. You might think that it's only plagiarism if you copy a term paper off the Internet. However, you can also plagiarize in a logic course, e.g., by copying a proof verbatim from the textbook (and only making the necessary changes to apply it to the assigned problem.) The point of logic problems which are similar to the proofs in the text is to make you work through those proofs, understand them, and then prove a similar result on the homework. Hence, all homework solutions must be in your own words; copying or paraphrasing closely from the text will be treated as plagiarism and results in a failing grade in the course, and a report to the Dean's office.

Checking your grades and reappraisals of work. University policies for reappraisal of term work and final grades apply (see the *Calendar* section "Reappraisal of Grades and Academic Appeals"). In particular, term work (homework assignments, midterms) will only be reappraised within 15 days of the date you are advised of your marks. Please keep track of your assignments (make sure to pick them up in lecture or in office hours) and your marks (check them on the website) and compare them with the graded work returned to you.

Exams. The midterm will be closed-book, and conducted in class (75 minutes). The final exam will be a cumulative take-home exam. There will be no collaboration on the final exam. Be aware that cheating on an exam is a serious academic offense and can result in suspension or expulsion.

Course Website

A course website on U of C's BlackBoard server has been set up. You should be automatically registered on the first day of class if you're registered in the class. You can find the website at http://blackboard.ucalgary.ca/

Log in with your UCS user ID and password. (You have a UCS ID if you have a ucalgary.ca email address. The ID is the part before the @; your password is the same password you use to check your email. If it works on webmail.ucalgary.ca, it should work on the BlackBoard server.) **You must log on at least once by the end of the second week of class.**

If you are not registered in the course on the first day of class, you will be added to the website as soon as you register, provided you have a UCS account. If you don't, you have to get one. You can register for one online at http://www.ucalgary.ca/it/register/. If you have forgotten your password, you will have to go to the IT Help Desk on the 7th floor of Math Sciences.

Tentative Syllabus and Due Dates

This is a tentative syllabus to give you a rough idea what parts of the book we will cover when.

Week 1: Recursive functions lectures 1–2 (Jan 13, 15). Chapter 6.

Learning goals: Understanding primitive recursion; constructing primitive recursive definitions; understanding minimization.

Week 2: Recursive and semi-recursive sets and relations lectures 3–4 (Jan 20, 22): Chapter 7.

Learning goals: Understanding decidability and recursive enumerability of sets and relations; relationships between recursive and semi-recursive sets and functions.

Diagnostic Assignment due Tuesday, Jan 20 (covers material from Phil 379).

Week 3: Equivalence of recursion and Turing computability lectures 5–6 (Jan 27, 29): Chapter 8.

Learning goals: Review of Turing machines and the Church-Turing thesis; coding Turing machine computations, carrying out primitive recursion by Turing machines.

Week 4: Arithmetization lectures 7–8 (Feb 3, 5). Chapter 15.

Learning goals: Understanding Gödel numbering; understanding recursion-theoretic properties of sets of formulas as those of the corresponding sets of Gödel numbers.

Assignment 1 due Thursday, Feb 5 (covers Chapters 6–8)

Week 5: Review of Logic and Model Theory of Arithmetic lecture 9–10 (Feb 10, 12). Chapters 9, 10, 12, 16.

Learning goals: Reviewing first-order logic; compactness and Löwenheim-Skolem theorems from PHIL 379. Understanding some basic arithmetical theories (Q, R), their relationships. Beginning nonstandard models of arithmetic.

Week 6: Representability in Q lecture 11–12 (Feb 24, 26): Chapter 16.

Learning goals: Understanding the concept of definability of a function in an arithmetical theory. Proving that all recursive functions are representable in Q.

Week 7: Incompleteness lectures 13–14 (Mar 2, 4). Chapter 17

Learning goals: Understanding the proof of the Diagonal Lemma. Applying the diagonal lemma to obtain Tarski's Theorem and Gödel's First Incompleteness Theorem.

Assignment 2 due Thursday, Mar 11 (covers Ch. 15–16)

Week 8: The Unprovability of Consistency lectures 15–16 (Mar 9, 11): Chapter 18

Learning goals: Understanding formalized consistency statements and provability conditions. The second incompleteness theorem.

Week 9: Catchup, midterm review lecture 17 (Mar 16)

Midterm Exam lecture 18: Thursday, Mar 18 (covers Chapters 6–8, 12, 15–17)

Week 10: Second-order Logic lectures 19–20 (Mar 23, 25): Chapter 22.

Learning goals: Understanding quantification over sets; expressive power of 2nd order logic; failure of compactness, Löwenheim-Skolem, completeness for 2nd order logic; 2nd order arithmetic.

Assignment 3 due Thursday, Mar 25 (covers Chapters 17, 18)

Week 11: Decidable cases of undecidable problems (1) lectures 20–21 (Mar 30, Apr 1): Chapter 21.

Learning goals: Understanding why monadic first and second-order logic are decidable.

Week 12: Decidable cases of undecidable problems (II) lectures 22–23 (Apr 6, 8): Chapter 24.

Learning goals: Understanding Presburger Arithmetic and why it is decidable.

Week 13: Catchup, Review lectures 24, 25 (Apr 13, 15).

Learning goals: The "big picture."

Assignment 4 due Thursday, Apr 15 (covers Chapters 21, 22, 24)

Final exam due Tuesday, Apr 20.