GNSS Ambiguity Bootstrapping: Theory and Application

P.J.G. Teunissen
Department of Mathematical Geodesy and Positioning
Delft University of Technology
Thijsseweg 11
2629 JA Delft, The Netherlands
Fax: ++ 31 15 278 3711
Email: P.J.G.Teunissen@geo.tudelft.nl

BIOGRAPHY

Peter Teunissen is Professor of Mathematical Geodesy and Positioning at the Delft University of Technology. He is currently engaged in the development of GNSS data processing strategies for medium scaled networks, with an emphasis on ambiguity resolution.

ABSTRACT

The purpose of carrier phase ambiguity resolution is to improve upon the precision of the estimated GNSS baseline by means of the integer ambiguity constraints. There exists a whole class of integer ambiguity estimators from which one can choose. Members from this class are, for instance, integer rounding, integer bootstrapping and integer least- squares. In this review paper we will present the theory and application of the method of integer bootstrapping. Particular emphasis is given to the probabilistic properties of the bootstrapped estimator. We will present the probability mass function of the bootstrapped estimator, the bootstrapped pull-in regions, the distribution of the bootstrapped baseline and easy-to-compute ways of evaluating the confidence regions of the GNSS baselines. They are important if one wants to perform a rigorous quality control.

INTRODUCTION

Global Navigation Satellite System (GNSS) ambiguity resolution is the process of resolving the unknown cycle ambiguities of double difference (DD) carrier phase data as integers. Ambiguity resolution applies to a great variety of GNSS models which are currently in use in navigation, surveying, geodesy and geophysics. An overview of these models, together with their applications, can be found in textbooks such as [*Hofmann-Wellenhof*, 1997], [*Leick*, 1995], [*Parkinson and Spilker*, 1996], [*Strang and Borre*, 1997], and [*Teunissen and Kleusberg*, 1998].

Any GNSS model can be cast in the following system of linear(ized) observation equations

$$y = Aa + Bb + e \tag{1}$$

where y is the given GNSS data vector of order m, a and b are the unknown parameter vectors respectively of order n and p, and where e is the noise vector. The data vector y will usually consist of the 'observed minus computed' single-, dual- or triple-frequency double-difference (DD) phase and/or pseudorange (code) observations accumulated over all observation epochs. The entries of vector a are then the DD carrier phase ambiguities, expressed in units of cycles rather than range. They are known to be *integers*, $a \in Z^n$. The entries of the vector b will consist of the remaining unknown parameters, such as for instance baseline components (coordinates) and possibly atmospheric delay parameters (troposphere, ionosphere). They are known to be real- valued, $b \in R^p$. Although vector b may contain other or more real-valued unknown parameters than only those of the baseline(s), we will in this contribution, as a matter of terminology, still call its estimator the baseline estimator.

The procedure which is usually followed for solving the GNSS model (1), can be divided into three steps [*Teunissen*, 1993]. In the first step one simply disregards the integer constraints $a \in Z^n$ on the ambiguities and performs a standard adjustment. As a result one obtains the (real-valued) estimates of a and b, together with their variance-covariance (vc-) matrix

$$\begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix}, \begin{bmatrix} Q_{\hat{a}} & Q_{\hat{a}\hat{b}} \\ Q_{\hat{b}\hat{a}} & Q_{\hat{b}} \end{bmatrix} \tag{2}$$

This solution is referred to as the 'float' solution. In the second step the 'float' ambiguity estimate $\hat{a}$ is used to compute the corresponding *integer* ambiguity estimate

$$\check{a} = S(\hat{a}) \tag{3}$$

with $S : R^n \mapsto Z^n$ a mapping from the n-dimensional space of real numbers to the n-dimensional space of integers. Once the integer ambiguities are computed, they are used in the third and final step to correct the 'float' estimate of b. As a result one obtains the ambiguity resolved baseline solution

$$\check{b} = \hat{b} - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1} (\hat{a} - \check{a}) \tag{4}$$

This solution is usually referred to as the 'fixed' baseline. The quality of the estimator $\check{b}$ depends on the quality of the 'float' solution, $\hat{a}$ and $\hat{b}$, and on the quality of the integer estimator $\check{a}$. When evaluating the quality of the GNSS baseline, one usually relies on the assumption that the integer ambiguities are deterministic. Strictly speaking this is not correct, as was pointed out in [*Teunissen*, 1990]. The integer ambiguities are estimated from the data and since the data are modelled as random variates, the estimated ambiguities are random variates too. They have their own probability distribution, despite their integerness. For a proper evaluation of the quality of the GNSS baseline, one should therefore take the random characteristics of the estimated integer ambiguities into account as well. Hence, we also need the probability distribution of the integer estimator $\check{a}$. This distribution depends however on the type of integer estimator chosen. Different choices of the map $S : R^n \mapsto Z^n$, will result in different integer estimators and will thus also produce differences in the probability distribution. In this contribution we will concentrate on the bootstrapping principle of integer estimation.

It is the purpose of the present contribution to review our current knowledge of the theory of integer bootstrapping. For easy reference the main results are formulated as theorems and corollaries. Proofs of these theorems and corollaries can be found in [*Teunissen*, 1998a+b, 1999a+b, 2001]. This contribution is organized as follows. In section 2 we introduce the bootstrapped ambiguity estimator and show how this integer estimator is related to sequential conditional least-squares and to the unique triangular decomposition of the ambiguity variance-covariance matrix. The link with the triangular decomposition is used to describe the bootstrapped pull-in regions and to show that the bootstrapped estimator is an admissible integer estimator. A probabilistic description of integer bootstrapping is given in section 3. Exact and closed form expressions are given for the bootstrapped probability mass function and for the distribution of the ambiguity bootstrapped baseline. These results enable one to study and evaluate the probabilistic properties of the bootstrapped baseline rigorously. We also present easy-to-compute measures for the bootstrapped baseline's probability of concentration. Applications of the bootstrapped estimator are presented and discussed in section 4.

INTEGER BOOTSTRAPPING

Sequential conditional least-squares

To prepare for our discussion of the bootstrapped estimator, we first consider the adjustment principles of 'conditional least-squares' and 'sequential conditional least-squares'. These principles form the basis of the bootstrapped estimator. We will show how the sequential conditional least-squares ambiguity estimator is constructed and how it is related to the unique lower triangular decomposition of the ambiguity vc-matrix. We commence with the principle of conditional least-squares estimation. The following corollary is a well known result from standard adjustment theory.

Corollary 1 (*Conditional least-squares*)

Let the expectation and dispersion of $\hat{a}_I = (\hat{a}_1, \ldots, \hat{a}_{i-1})^T \in R^{i-1}$ and $\hat{a}_i \in R$ be given as

$$E\{\begin{bmatrix} \hat{a}_I \\ \hat{a}_i \end{bmatrix}\} = \begin{bmatrix} a_I \\ a_i \end{bmatrix} \;,\; D\{\begin{bmatrix} \hat{a}_I \\ \hat{a}_i \end{bmatrix}\} = \begin{bmatrix} Q_I & Q_{Ii} \\ Q_{iI} & \sigma_i^2 \end{bmatrix}$$

Then the least-squares estimator of a_i, when a_I is constrained to the fixed vector z_I, is given as

$$\hat{a}_{i|I} = \hat{a}_i - Q_{iI} Q_I^{-1} (\hat{a}_I - z_I) \tag{5}$$

The estimator $\hat{a}_{i|I}$ is referred to as the conditional least-squares ambiguity estimator. It is conditioned on fixing the previous ambiguities to the values z_j, $j = 1, \ldots, (i-1)$. Note that $\hat{a}_{i|I}$ and $\hat{a}_I$ are uncorrelated. This is an important property that will be used repeatedly in the following.

The above result can be used to derive a sequential version of the conditional least-squares estimator. For $i = 2$, we obtain the scalar version of (5)

$$\hat{a}_{2|1} = \hat{a}_2 - \sigma_{21} \sigma_1^{-2} (\hat{a}_1 - z_1) \tag{6}$$

in which $\hat{a}_{2|1}$ is uncorrelated with $\hat{a}_1$. For $i = 3$, the conditional least-squares estimator $\hat{a}_{3|2,1}$ follows from fixing the two ambiguities a_1 and a_2 to the values z_1 and z_2. Note however, since $\hat{a}_{3|2,1}$ is invariant to any regular transformation of $\hat{a}_1, \hat{a}_2$, that we may as well fix $\hat{a}_1$ and $\hat{a}_{2|1}$ to the values z_1 and z_2. This has the advantage that matrix Q_I of (5) becomes diagonal. As a result we obtain

$$\hat{a}_{3|2,1} = \hat{a}_3 - \sigma_{3,1} \sigma_1^{-2} (\hat{a}_1 - z_1) - \sigma_{3,2|1} \sigma_{2|1}^{-2} (\hat{a}_{2|1} - z_2) \tag{7}$$

in which $\hat{a}_{3|2,1}$ is uncorrelated with both $\hat{a}_1$ and $\hat{a}_{2|1}$. It will be clear, that we may continue in this way to obtain the corresponding expressions for the next and following ambiguities as well. The result is summarized in the following corollary.

Corollary 2 (*Sequential conditional least-squares*)

The conditional least-squares estimator $\hat{a}_{i|I}$ can be computed sequentially as

$$\hat{a}_{i|I} = \hat{a}_i - \sum_{j=1}^{i-1} \sigma_{i,j|J} \sigma_{j|J}^{-2} (\hat{a}_{j|J} - z_j) \;,\; i = 1, \ldots, n \tag{8}$$

where $\sigma_{i,j|J}$ denotes the covariance between $\hat{a}_i$ and $\hat{a}_{j|J}$, and $\sigma_{j|J}^2$ is the variance of $\hat{a}_{j|J}$. For $i = 1$, $\hat{a}_{i|I}$ is set equal to $\hat{a}_1$.

With this result we are now also in a position to show how sequential conditional least-squares estimation relates to the unique lower triangular decomposition of the ambiguity vc-matrix. From (8) it follows that the difference $(\hat{a}_i - z_i)$ may be written in terms of the differences $(\hat{a}_{j|J} - z_j)$, $j = 1, \ldots, i$, as

$$(\hat{a}_i - z_i) = (\hat{a}_{i|I} - z_i) + \sum_{j=1}^{i-1} \sigma_{i,j|J} \sigma_{j|J}^{-2} (\hat{a}_{j|J} - z_j) \tag{9}$$

When written out in vector-matrix form, this gives

$$\begin{bmatrix} \hat{a}_1 - z_1 \\ \hat{a}_2 - z_2 \\ \vdots \\ \hat{a}_n - z_n \end{bmatrix} = \begin{bmatrix} 1 & & & \\ l_{21} & 1 & & \\ \vdots & \vdots & \ddots & \\ l_{n1} & l_{n2} & \dots & 1 \end{bmatrix} \begin{bmatrix} \hat{a}_1 - z_1 \\ \hat{a}_{2|1} - z_2 \\ \vdots \\ \hat{a}_{n|N} - z_n \end{bmatrix} \quad (10)$$

with $l_{ij} = \sigma_{i,j|J}\sigma_{j|J}^{-2}$, for $1 \leq j < i \leq n$. Since the sequential conditional least-squares ambiguities are mutually uncorrelated, their vc-matrix is diagonal, as a consequence of which the vc-matrix of the $\hat{a}_i$ is given a triangular decomposition, when applying the error propagation law to (10). The relation between sequential conditional least-squares and the triangular decomposition is summarized in the following corollary.

Corollary 3 (*The statistics of the triangular decomposition*)

Let the $\hat{a}_i$, $i = 1, \dots, n$, be collected in the vector $\hat{a} = (\hat{a}_1, \dots, \hat{a}_n)^T$ and let their sequential conditional least-squares estimators be collected in the vector $\hat{a}_c = (\hat{a}_1, \dots, \hat{a}_{n|N})^T$. Then $\hat{a}$ and $\hat{a}_c$, and their vc-matrices, are related as

$$\hat{a} - z = L(\hat{a}_c - z) \text{ and } Q_{\hat{a}} = LDL^T \quad (11)$$

where the matrix entries are given as

$$(L)_{ij} = \begin{cases} 0 & \text{for} \quad 1 \leq i < j \leq n \\ 1 & \text{for} \quad i = j \\ \sigma_{i,j|J}\sigma_{j|J}^{-2} & \text{for} \quad 1 \leq j < i \leq n \end{cases}$$

$$\text{and } D = \text{diag}(\dots, \sigma_{j|J}^2, \dots)$$

The bootstrapped estimator

We are now in a position to describe the integer bootstrapping principle. In order to compute the sequential conditional least-squares solutions, one needs to specify the z_j on which the conditioning takes place. In case of bootstrapping, z_j, for $j = 1, \dots, n$, is chosen as the nearest integer of $\hat{a}_{j|J}$. Hence, for $\hat{a}_{i|I}$ the conditioning takes place on the nearest integers of all previous $i-1$ conditional estimates. The ith component of the bootstrapped solution itself is then given as the nearest integer of $\hat{a}_{i|I}$. We thus have the following definition.

Definition (*Integer bootstrapping*)

Let $\hat{a} = (\hat{a}_1, \dots, \hat{a}_n)^T \in R^n$ be the ambiguity 'float' solution and let $\check{a}_B = (\check{a}_{B,1}, \dots, \check{a}_{B,n})^T \in Z^n$ denote the corresponding integer bootstrapped solution. The entries of the bootstrapped ambiguity estimator are then defined as

$$\begin{aligned} \check{a}_{B,1} &= [\hat{a}_1] \\ \check{a}_{B,2} &= [\hat{a}_{2|1}] = [\hat{a}_2 - \sigma_{21}\sigma_1^{-2}(\hat{a}_1 - \check{a}_{B,1})] \\ &\vdots \\ \check{a}_{B,n} &= [\hat{a}_{n|N}] = [\hat{a}_n - \textstyle\sum_{j=1}^{n-1} \sigma_{n,j|J}\sigma_{j|J}^{-2}(\hat{a}_{j|J} - \check{a}_{B,j})] \end{aligned} \quad (12)$$

where $'[.]'$ denotes the operation of rounding to the nearest integer.

As the definition shows, the bootstrapped estimator can be seen as a generalization of the method of 'integer rounding'. If n ambiguities are available, one starts with the first ambiguity $\hat{a}_1$ and rounds its value to the nearest integer. Having obtained the integer value of this first ambiguity, the real-valued estimates of all remaining ambiguities are then corrected by virtue of their correlation with the first ambiguity. Then the second, but now corrected, real-valued ambiguity estimate is rounded to its nearest integer. Having obtained the integer value of the second ambiguity, the real-valued estimates of all remaining $n-2$ ambiguities are then again corrected, but now by virtue of their correlation with the second ambiguity. This process is continued until all ambiguities are accommodated. Thus the bootstrapped estimator reduces to 'integer rounding' in case correlations are absent, i.e. in case the ambiguity vc-matrix is diagonal.

Note that the bootstrapped estimator is not unique. Changing the order in which the ambiguities appear in vector $\hat{a}$ will already produce a different bootstrapped estimator. Although the principle of bootstrapping remains the same, every choice of ambiguity parametrization has its own bootstrapped estimator.

The bootstrapped pull-in regions

Integer bootstrapping is not the only principle which one can use for estimating integer ambiguities. Two other principles are, for example, integer rounding and integer least-squares. In fact, there exists a whole class of integer estimators from which one can choose. In order to introduce this class, we start from the map $S : R^n \mapsto Z^n$. Due to the discrete nature of Z^n, the map S will not be one-to-one, but instead a many-to-one map. This implies that different real-valued ambiguity vectors will be mapped to the same integer vector. One can therefore assign a subset $S_z \subset R^n$ to each integer vector $z \in Z^n$:

$$S_z = \{x \in R^n \mid z = S(x)\}, \quad z \in Z^n \quad (13)$$

The subset S_z contains all real-valued ambiguity vectors that will be mapped by S to the same integer vector $z \in Z^n$. This subset is referred to as the *pull-in region* of z. It is the region from which all ambiguity 'float' solutions are pulled to the same 'fixed' ambiguity vector z. Since the pull-in regions define the integer estimator completely, one can define classes of integer estimators by imposing various conditions on the pull-in regions. The class of admissible integer ambiguity estimators is defined as follows [*Teunissen*, 1999b].

Definition (*Admissible integer estimators*)

The integer estimator $\check{a} = S(\hat{a})$ is said to be *admissible* when its pull-in regions $S_z = \{x \in R^n \mid z = S(x)\}$, $z \in Z^n$, satisfy

$$(i) \quad \cup_{z \in Z^n} S_z = R^n$$

$$(ii) \quad \text{Int}S_{z_1} \cap \text{Int}S_{z_2} = \emptyset,\ \forall z_1, z_2 \in Z^n, z_1 \neq z_2 \quad (14)$$

$$(iii) \quad S_z = z + S_0,\ \forall z \in Z^n$$

One can now show that the bootstrapped estimator belongs to this class of admissible integer estimators. In order to do so, we first need the bootstrapped pull-in regions. They are given in the following corollary.

Corollary 4 (*Bootstrapped pull-in regions*)

The pull-in regions of the bootstrapped ambiguity estimator $\check{a}_B = (\check{a}_{B,1}, \ldots, \check{a}_{B,n})^T \in Z^n$ are given as

$$S_{B,z} = \{x \in R^n \mid | c_i^T L^{-1}(x - z) | \leq \frac{1}{2}\ ,\ i = 1, \ldots, n\} \quad (15)$$

$\forall z \in Z^n$, where L denotes the unique unit lower triangular matrix of the ambiguity vc-matrix' decomposition $Q_{\hat{a}} = LDL^T$ and c_i denotes the ith canonical unit vector having a 1 as its ith entry and zeros otherwise.

That the bootstrapped estimator is indeed admissible, can now be seen as follows. The first two conditions of (14) are easily verified using the definition of the bootstrapped estimator. Since every real-valued vector $\hat{a}$ will be mapped by the bootstrapped estimator to an integer vector, the pull-in regions $S_{B,z}$ cover R^n without any gaps. There is also no overlap between the pull-in regions, since - apart from boundary ties - any real-valued vector $\hat{a}$ is mapped to not more than one integer vector. To verify the last condition of (14), we make use of (15). From

$$\begin{aligned} S_{B,z} &= \{x \in R^n \mid | c_i^T L^{-1}(x - z) | \leq \tfrac{1}{2}\ ,\ i = 1, \ldots, n\} \\ &= \{x \in R^n \mid | c_i^T L^{-1} y | \leq \tfrac{1}{2}\ ,\ x = y + z\ ,\ i = 1, \ldots, n\} \\ &= S_{B,0} + z \end{aligned}$$

it follows that all bootstrapped pull-in regions are translated copies of $S_{B,0}$. All pull-in regions have therefore the same shape and the same volume. Their volumes all equal 1. This can be shown by transforming $S_{B,0}$ to the unit cube centered at the origin. Consider the linear transformation $y = L^{-1}x$. Then

$$L^{-1}(S_{B,0}) = \{y \in R^n \mid | c_i^T y | \leq \frac{1}{2}\ ,\ i = 1, \ldots, n\}$$

equals the unit cube centered at the origin. Since the determinant of the unit lower triangular matrix L^{-1} equals one and since the volume of the unit cube equals one, it follows that the volume of $S_{B,0}$ must equal one as well. To infer the shape of the bootstrapped pull-in region, we consider the two-dimensional case first. Let the lower triangular matrix L be given as

$$L = \begin{bmatrix} 1 & 0 \\ l & 1 \end{bmatrix}$$

Then

$$\begin{aligned} S_{B,0} &= \{x \in R^2 \mid | c_i^T L^{-1} x | \leq \tfrac{1}{2}\ ,\ i = 1, 2\} \\ &= \{x \in R^2 \mid | x_1 | \leq \tfrac{1}{2}, | x_2 - l x_1 | \leq \tfrac{1}{2}\} \end{aligned}$$

which shows that the two-dimensional pull-in region equals a parallellogram. Its region is bounded by the two vertical lines $x_1 = \frac{1}{2}$ and $x_1 = -\frac{1}{2}$, and the two parallel slopes $x_2 = l x_1 + \frac{1}{2}$ and $x_2 = l x_1 - \frac{1}{2}$. The direction of the slope is governed by $l = \sigma_{21}\sigma_1^{-2}$. Hence, in the absence of correlation between the two ambiguities, the parallellogram reduces to the unit square. In higher dimensions the above construction of the pull-in region can be continued. In three dimensions for instance, the intersection of the pull-in region with the $x_1 x_2$-plane remains a parallellogram, while along the third axis the pull-in region becomes bounded by two parallel planes.

Figure 1 shows 3 examples of two-dimensional pull-in regions, namely of integer rounding, integer bootstrapping, and of integer least- squares. The shape of the ellipse as determined by the ambiguity vc-matrix is also shown. The bootstrapped pull-in region is a parallellogram, the pull-in region of rounding is a square and the pull-in region of integer least-squares is a hexagon. All three pull-in regions have the same area, namely one. The shapes of the pull-in regions of bootstrapping and least-squares are determined by the ambiguity vc-matrix. The shape of the pull-in region of rounding is however independent of the ambiguity vc-matrix. It always equals the unit-square.

BOOTSTRAPPED DISTRIBUTIONS

The bootstrapped probability mass function

In this section we present exact and closed-form expressions for the distribution of the bootstrapped ambiguity estimator and for the distribution of the ambiguity bootstrapped baseline. The distribution of an admissible ambiguity estimator can be determined once its pull-in regions are known and once the probability density function (pdf) of the 'float' solution is given. Since the integer estimator is by definition of the discrete type, its distribution will be a probability mass function (pmf). It has zero masses at non-integer points and nonzero masses at some or all integer points. The pmf of the integer bootstrapped estimator $\check{a}_B$ will be denoted as $P(\check{a}_B = z)$, with $z \in Z^n$. The pdf of the 'float' ambiguity solution $\hat{a}$ will be denoted as $p_{\hat{a}}(x)$.

Since the integer bootstrapped estimator is defined as $\check{a}_B = z \Longleftrightarrow \hat{a} \in S_{B,z}$, it follows that $P(\check{a}_B = z) = P(\hat{a} \in S_{B,z})$. The pmf of $\check{a}_B$ follows therefore as

$$P(\check{a}_B = z) = \int_{S_{B,z}} p_{\hat{a}}(x) dx\ ,\ z \in Z^n \quad (16)$$

Hence, the probability that $\check{a}_B$ coincides with z is given by the integral of the pdf $p_{\hat{a}}(x)$ over the bootstrapped pull-in region $S_{B,z} \subset R^n$.

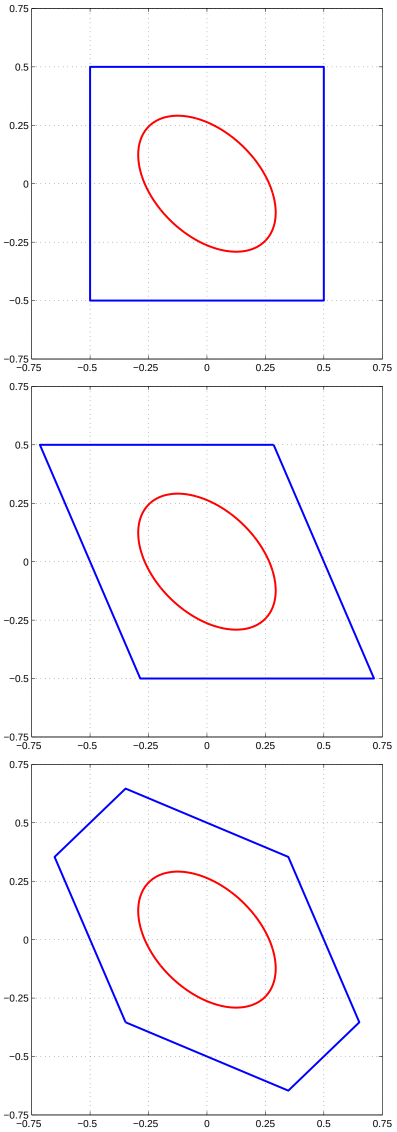

Fig. 1. The shape of the ambiguity ellipse and the two-dimensional pull-in regions of (top) integer rounding; (middle) integer bootstrapping; and (bottom) integer least-squares

Note that the above expression holds for any distribution the 'float' ambiguities $\hat{a}$ might have. In most GNSS applications however, one usually assumes the vector of observables y to be normally distributed. In that case the 'float' solutions $\hat{a}$ and $\hat{b}$, both being linear estimators, will be normally distributed too. In the remainder of this contribution we shall therefore assume that the 'float' solutions are normally distributed as

$$\begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix} \sim \mathrm{N}\left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} Q_{\hat{a}} & Q_{\hat{a}\hat{b}} \\ Q_{\hat{b}\hat{a}} & Q_{\hat{b}} \end{bmatrix} \right) \quad (17)$$

The following theorem, due to Teunissen (2001), gives an exact expression for the bootstrapped pmf in case (17) holds true. As the theorem shows, the pmf equals a product of univariate pmf's and is therefore very easy to compute.

Theorem 1 (*The integer bootstrapped pmf*)

Let $\hat{a}$ be distributed as $\mathrm{N}(a, Q_{\hat{a}})$, $a \in Z^n$, and let $\check{a}_B$ be the corresponding integer bootstrapped estimator. Then

$$P(\check{a}_B = z) = \prod_{i=1}^{n} [\Phi(\frac{1-2l_i^T(a-z)}{2\sigma_{\hat{a}_{i|I}}}) + \Phi(\frac{1+2l_i^T(a-z)}{2\sigma_{\hat{a}_{i|I}}}) - 1]\ ,\ z \in Z^n \quad (18)$$

with

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}v^2\} dv$$

and with l_i the ith column vector of the unit lower triangular matrix L^{-T} and $\sigma^2_{\hat{a}_{i|I}}$ the variance of the ith least-squares ambiguity obtained through a conditioning on the previous $I = \{1, \ldots, (i-1)\}$ ambiguities.

It follows from (18) that the bootstrapped pmf is symmetric about the mean of $\hat{a}$. This implies that the bootstrapped estimator $\check{a}_B$ is an unbiased estimator of $a \in Z^n$. Since the 'float' solutions, $\hat{a}$ and $\hat{b}$, are unbiased too, it follows from taking the expectation of (4) that the bootstrapped baseline is also unbiased.

For the purpose of predicting the success of ambiguity resolution, the probability of correct integer estimation is of particular interest. For the bootstrapped estimator this success rate is given in the following corollary.

Corollary 5 (*The bootstrapped success rate*)

The bootstrapped probability of correct integer estimation (the success rate) is given as

$$P(\check{a}_B = a) = \prod_{i=1}^{n} [2\Phi(\frac{1}{2\sigma_{\hat{a}_{i|I}}}) - 1] \quad (19)$$

From (18) it follows that the bootstrapped pmf reaches its maximum at its point of symmetry. Thus $\max_z P(\check{a}_B = z) = P(\check{a}_B = a)$. This is a reassuring result, since its implies that the bootstrapped success rate is largest of all the bootstrapped probability masses.

Finally observe that the shape of the bootstrapped pmf is completely governed by the ambiguity vc-matrix $Q_{\hat{a}}$. The pmf follows once the triangular factor L and the diagonal matrix D of the decomposition $Q_{\hat{a}} = LDL^T$ are given.

The distribution of the bootstrapped baseline

We are now in the position to determine the pdf of the bootstrapped baseline

$$\check{b}_B = \hat{b} - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1} (\hat{a} - \check{a}_B) \quad (20)$$

In order to determine the pdf of this baseline estimator, one needs to propagate the uncertainty of the 'float' solution, $\hat{a}$ and $\hat{b}$, as well as the uncertainty of the integer solution $\check{a}_B$ through (20). Should one neglect the random character of the integer solution and therefore consider the ambiguity vector $\check{a}_B$ as deterministic and equal to, say, z, then the pdf of $\check{b}_B$ would equal the conditional baseline distribution

$$p_{\hat{b}|\hat{a}}(x \mid y = z) = \frac{1}{\sqrt{\det Q_{\hat{b}|\hat{a}}}(2\pi)^{\frac{1}{2}p}} \exp\{-\tfrac{1}{2} \parallel x - b_{|\hat{a}=z} \parallel^2_{Q_{\hat{b}|\hat{a}}}\} \qquad (21)$$

with conditional mean $b_{|\hat{a}=z} = b - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1}(a - z)$, conditional variance matrix $Q_{\hat{b}|\hat{a}} = Q_{\hat{b}} - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1} Q_{\hat{a}\hat{b}}$ and $\parallel . \parallel^2_{Q_{\hat{b}|\hat{a}}} = (.)^T Q_{\hat{b}|\hat{a}}^{-1}(.)$. However, since $\check{a}_B$ is random and not deterministic, the conditional baseline distribution will give a too optimistic description of the quality of the 'fixed' baseline. To get a correct description of the 'fixed' baseline's pdf, the integer ambiguity's pmf needs to be considered. As the following theorem shows this results in a baseline distribution, which generally will be multi-modal.

Theorem 2 (*Distribution of the bootstrapped baseline*)

Let the 'float' solution, $\hat{a}$ and $\hat{b}$, be distributed as in (17), let $\check{a}_B$ be the integer bootstrapped estimator and let the 'fixed' bootstrapped baseline $\check{b}_B$ be given as in (20). The pdf of $\check{b}_B$ reads then

$$p_{\check{b}_B}(x) = \sum_{z \in Z^n} p_{\hat{b}|\hat{a}}(x \mid y = z) P(\check{a}_B = z) \qquad (22)$$

This theorem was first introduced and proved in [*Teunissen*, 1999a]. Note that, although the model (1) is linear and the observables normally distributed, the distribution of the 'fixed' baseline is not normal, but multi-modal. As the theorem shows, the 'fixed' baseline distribution equals an infinite sum of weighted conditional baseline distributions. These conditional baseline distributions $p_{\hat{b}|\hat{a}}(x \mid y = z)$ are shifted versions of one another. The size and direction of the shift is governed by $Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1} z$, $z \in Z^n$. Each of the conditional baseline distributions in the infinite sum is downweighted. These weights are given by the probability masses of the distribution of the integer bootstrapped ambiguity estimator $\check{a}_B$. This shows that the dependence of the 'fixed' baseline distribution on the choice of integer estimator is only felt through the weights $P(\check{a}_B = z)$.

Figure 2 shows two examples of the multi-modality of the distribution of the 'fixed' DD range in case of single-epoch ambiguity resolution, based on the long baseline (ionosphere-float) geometry-free GPS model. Figure 2a shows the distribution in case dual-frequency data are used, while figure 2b corresponds to the triple-frequency case (modernised GPS).

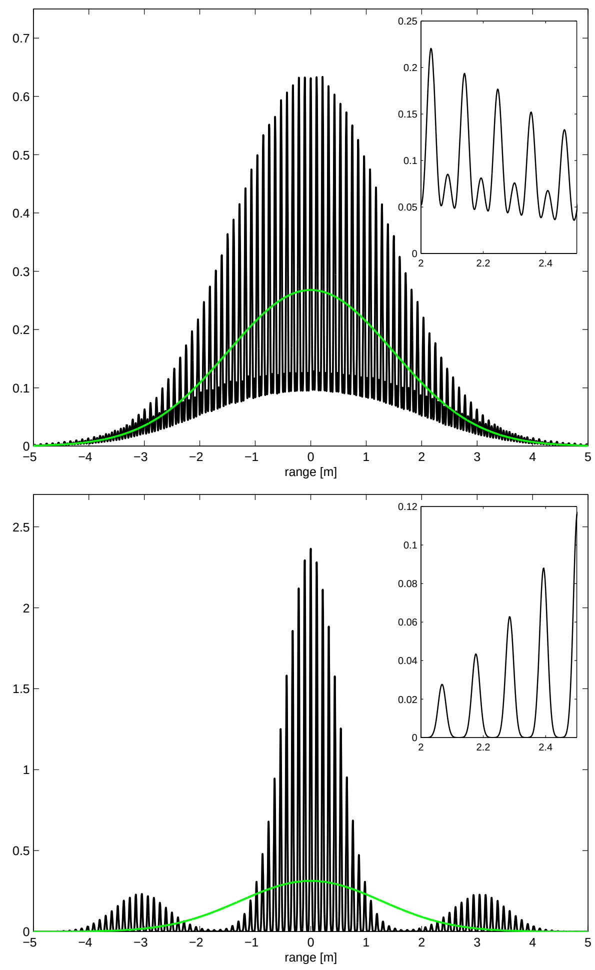

Fig. 2. Distribution of the 'fixed' DD range in case of single-epoch ambiguity resolution, based on the long baseline (ionosphere-float) geometry-free GPS model: (a, top) dual-frequency case; (b, bottom) triple-frequency case. The uni-modal normal distributions shown in green are those of the corresponding "float" solution for the DD range.

The quality of the bootstrapped baseline

In order to describe the quality of the bootstrapped baseline, one would like to know how close one can expect the baseline estimate $\check{b}_B$ to be to the unknown, but true baseline value b. As a measure of confidence, we take

$$P(\check{b}_B \in R) = \int_R p_{\check{b}_B}(x)dx \;\text{ with }\; R \subset R^p \qquad (23)$$

But in order to evaluate this integral, we first need to make a choice about the shape and location of the subset R. Since it is common practice in GNSS positioning to use the vc-matrix of the conditional baseline estimator as a measure of precision for the 'fixed' baseline, the vc-matrix $Q_{\hat{b}|\hat{a}}$ will be used to define the shape of the confidence region. For its location, we choose the confidence region to be centered at b. After all, we would like to know by how much the baseline estimate $\check{b}_B$ can be expected to differ from the true, but unknown baseline value b. That is, one would like (23) to be a measure of the bootstrapped baseline's probability of concentration about b.

With these choices on shape and location, the confidence region R takes the form

$$R = \{x \in R^p \mid (x - b)^T Q_{\hat{b}|\hat{a}}^{-1}(x - b) \;\leq\; \beta^2\} \qquad (24)$$

The size of the region can be varied by varying β.

The following theorem shows how the bootstrapped baseline's probability of concentration (23) can be evaluated as a weighted sum of probabilities of noncentral Chi-square distributions [*Teunissen,* 1999a].

Theorem 3 (*The bootstrapped baseline's probability of concentration*)

Let $\check{b}_B$ be the ambiguity bootstrapped baseline estimator, R be defined as in (24), and $\chi^2(p,\lambda_z)$ denote the noncentral Chi-square distribution with p degrees of freedom and noncentrality parameter λ_z. Then

$$P(\check{b}_B \in R) = \sum_{z \in Z^n} P(\chi^2(p,\lambda_z) \leq \beta^2) P(\check{a}_B = z) \qquad (25)$$

with

$$\lambda_z = \| \nabla\check{b}_z \|^2_{Q_{\hat{b}|\hat{a}}} \quad \text{and} \quad \nabla\check{b}_z = Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1}(z-a)$$

This result shows that the probability of the ambiguity bootstrapped baseline lying inside the ellipsoidal region R centered at b equals an infinite sum of probability products. If one considers the two probabilities of these products separately, two effects are observed. First the probabilistic effect of shifting the conditional baseline estimator away from b and secondly the probabilistic effect of the peakedness or nonpeakedness of the bootstrapped pmf. The second effect is related to the expected performance of bootstrapped ambiguity resolution, while the first effect has to do with the sensitivity of the baseline for changes in the values of the integer ambiguities. This effect is measured by the noncentrality parameter λ_z. Since the tail of a noncentral Chi-square distribution becomes heavier when the noncentrality parameter increases, while the degrees of freedom remain fixed, $P(\chi^2(p,\lambda_z) \leq \beta^2)$ gets smaller when λ_z gets larger.

The two probabilities in the product reach their maximum values when $z = a$. The following corollary shows how these two maxima can be used to lower bound and to upper bound the probability $P(\check{b}_B \in R)$. Such bounds are of importance for practical purposes, since it is difficult in general to evaluate (25) exactly.

Corollary 6 (*Lower and upper bounds*)

Let $\check{b}_B$ be the ambiguity bootstrapped baseline estimator and let R be defined as in (24). Then

$$P(\hat{b}_{|\hat{a}=a} \in R) P(\check{a}_B = a) \;\leq P(\check{b}_B \in R) \;\leq P(\hat{b}_{|\hat{a}=a} \in R) \qquad (26)$$

with

$$\begin{cases} P(\hat{b}_{|\hat{a}=a} \in R) & = & P(\chi^2(p,0) \leq \beta^2) \\ P(\check{a}_B = a) & = & \prod_{i=1}^{n} [2\Phi(\frac{1}{2\sigma_{\hat{a}_{i|I}}}) - 1] \end{cases}$$

Note that the two bounds relate the probability of the bootstrapped baseline estimator to that of the conditional estimator and to the bootstrapped succes rate. The above bounds become tight when the ambiguity success rate approaches one. This shows, although the probability of the conditional estimator always overestimates the probability of the bootstrapped baseline estimator, that the two probabilities are close for large values of the success rate. This implies that in case of GNSS ambiguity resolution, one should first evaluate the bootstrapped success rate $P(\check{a}_B = a)$ and make sure that its value is close enough to one, before making any inferences on the basis of the distribution of the conditional baseline estimator. In other words, the (unimodal) distribution of the conditional estimator is a good approximation to the (multimodal) distribution of the bootstrapped baseline estimator, when the success rate is sufficiently close to one.

APPLICATIONS OF INTEGER BOOTSTRAPPING

Bootstrapping as a genuine ambiguity resolver

In the previous sections we have presented the theory of integer bootstrapping and showed that the principle of bootstrapping can be used as a viable option for resolving the integer carrier phase ambiguities. The method of integer bootstrapping is easy to implement and it does not need, as opposed to the method of integer least-squares, an integer search for computing the sought for integer solution. However, as it was mentioned earlier, the outcome of bootstrapping depends on the chosen ambiguity parametrization. Bootstrapping of DD ambiguities, for instance, will produce an integer solution which generally differs from the integer solution obtained from bootstrapping of reparametrized ambiguities. Since this dependency also holds true for the bootstrapped pmf, one still has some important degrees of freedom left for improving (25) or for sharpening the lower bound of (26).

In order to improve the bootstrapped success rate, one should work with decorrelated ambiguities instead of with the original ambiguities. The method of bootstrapping performs relatively poor, for instance, when applied to the DD ambiguities. This is due to the usually high correlation between the DD ambiguities. Bootstrapping should therefore be used in combination with the decorrelating Z-transformation of the LAMBDA method. This transformation decorrelates the ambiguities further than the best reordering would achieve and thereby reduces the values of the sequential conditional variances. By reducing the values of the sequential conditional variances, the bootstrapped success rate gets enlarged.

Bootstrapping in the context of integer least-squares

The estimation principles of integer bootstrapping and integer least- squares both result in integer estimators which are admissible. The following theorem, due to Teunissen (1999b), shows however that integer bootstrapping is outperformed by integer least-squares.

Theorem 4 (*Integer least-squares is optimal*)

Let the integer least-squares estimator be given as $\check{a}_{LS} = \arg\min_{z \in Z^n} \| \hat{a} - z \|^2_{Q_{\hat{a}}}$. Then

$$P(\check{a}_{LS} = a) \geq P(\check{a} = a) \tag{27}$$

for any admissible estimator $\check{a}$.

This result shows that the integer least-squares estimator is best in the sense that it maximizes the ambiguity success rate. When aiming at a large as possible success rate, one is thus better of using the integer least-squares estimator than any other admissible estimator, including the bootstrapped estimator. Does this result automatically disqualify the applicability of integer bootstrapping? No, not quite. From theorem 4 it follows that

$$P(\check{a}_B = a) \leq P(\check{a}_{LS} = a) \tag{28}$$

A very useful application of this result is now that it shows how one can *lower-bound* the probability of correct integer least-squares estimation. This is particularly useful since the success rate of integer least-squares is usually difficult to compute, whereas the bootstrapped success rate is very easy to compute. Moreover, when the above lower bound is applied to the decorrelated ambiguities, it becomes a very sharp lower bound. In fact, the bootstrapped lower bound is presently the best available lower bound of the least-squares success rate. This has also been verified empirically by various researchers, and more recently also by Thomson (2000).

The fact that the bootstrapped estimator is so easy to compute and still gives a good approximation to the integer least-squares estimator when applied to the decorrelated ambiguities, makes it also an important tool for setting the size of the ambiguity search space. Following the decorrelation step, the LAMBDA-method uses, as one of its options, the bootstrapped solution $\check{a}_B$ for setting the size of the ambiguity search space as

$$(\hat{a} - z)^T Q_{\hat{a}}^{-1} (\hat{a} - z) \leq \chi^2$$

with

$$\chi^2 = (\hat{a} - \check{a}_B)^T Q_{\hat{a}}^{-1} (\hat{a} - \check{a}_B)$$

In this way one can work with a very small search space and still guarantee that the sought for integer least-squares solution is contained in it. For more information on the LAMBDA method, the reader is referred to [*Teunissen*, 1993], [*Teunissen*, 1995] and [*de Jonge and Tiberius*, 1996a] or to the textbooks [*Hofmann-Wellenhof*, 1997], [*Strang and Borre*, 1997], [*Teunissen and Kleusberg*, 1998]. Practical results obtained with it can be found, for example, in [*Boon and Ambrosius*, 1997], [*Boon et al.*, 1997], [*Cox and Brading*, 1999], [*de Jonge and Tiberius*, 1996b], [*de Jonge et al.*, 1996], [*Han*, 1995], [*Jonkman*, 1998], [*Peng et al.*, 1999], [*Tiberius and de Jonge*, 1995], [*Tiberius et al.*, 1997].

Bootstrapping for partial ambiguity resolution

One usually aims at resolving all ambiguities simultaneously. It may happen however that it is simply not possible to resolve the complete vector of ambiguities with sufficient probability. As an alternative of resolving the complete vector of ambiguities, one might then consider resolving only a subset of the ambiguities. This idea of *partial* ambiguity resolution was introduced in [*Teunissen et al.*, 1999], where it was applied to long baselines using the current GPS. The idea of partial ambiguity resolution is based on the fact that the success rate will generally increase when fewer integer constraints are imposed. However, in order to apply partial ambiguity resolution, one first will have to determine which subset of ambiguities to choose. It will be clear that this decision should be based on the precision of the 'float' ambiguities. The more precise the ambiguities, the larger the ambiguity success rate. It is at this point where the decorrelation step of the LAMBDA method and the bootstrapping principle can be applied. Once the transformed and decorrelated ambiguity vc-matrix is obtained, the construction of the subset proceeds in a sequential fashion. One first starts with the most precise ambiguity, say $\hat{z}_1$, and computes its success rate $P(\check{z}_1 = z_1)$. If this success rate is large enough, one continues and determines the most precise pair of ambiguities, say $(\hat{z}_1, \hat{z}_2)$. If their success rate is still large enough, one continues again by trying to extend the set. This procedure continues until one reaches a point where the corresponding success rate becomes unacceptably small. When this point is reached, one can expect that the previously identified ambiguities can be resolved successfully.

Once the subset for partial ambiguity resolution has been identified, one still needs to determine what this will do to improve the baseline estimator. After all, being able to successfully resolve the ambiguities does not necessarily mean that the 'fixed' solution is significantly better than the 'float' solution. The theory presented in the previous sections provide the necessary tools for performing such an evaluation rigorously.

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